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INNOMATH

Innovative enriching education processes for Mathematically Gifted Students in Europe

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Learning Plan

Topic: Linear Programming - The simplex algorithm

Target Group:

"Gifted" Students in a high school at grade level 12 or 11 (4th year in a secondary school)
Mathematical background of the students:
Basic Concepts of Algebra,
Functions, Graphs of linear functions,
Solving linear equations and systems of equations,
Solving linear inequalities.

Goal/ Content/ Description:

- To help students understand that most linear programming problems have many variables, which means that the graphical solution method cannot be applied to linear programming problems with more than two variables, we will introduce students to the Simplex algorithm used.
- To solving a large number of industrial applications, where it is required to solve problems with thousands of restrictions and variables.

Objectives:

General Mathematical Objectives

bringing the problem to the standard form;
transformation of the standard form into a tabular form;
solving the problem using the simplex method

What the students will be able to do:
to bring the problem to the standard form;
to transform the standard form into a tabular form;
solve the problem using the simplex method
To develop problem solving skills;
To identify, develop and create applications of related concepts and processes in the real world.

Materials/ Tools:

- Computers and or scientific calculators
- Graphing software (Google drawings etc)
- Spreadsheet
- Power Point Presentations

Resources used by the teacher:

- Introductory books on linear programming
- Examples, exercises, ppt presentations, YouTube videos on the topic by using the Internet.
- School Textbooks covering the topic

Resources for the student:

- Examples, exercises, ppt presentations, YouTube videos on the topic by using the Internet.
- School Textbooks covering the topic.
- Work sheets prepared by the teacher

Approaches/ Methodology:

In 1947, George Dantzig developed an efficient method for solving linear programming problems, called the simplex algorithm. The algorithm involves a search through the set of possible solutions to find the optimal solution to the proposed objective. Since the development of the simplex algorithm, linear programming is used in all fields from industry to banking, education, forestry, oil leading to the solution of a wide variety of problems: the establishment of human resources, the operations of hydropower systems, supply machine routes, etc. Danzing's method, with some relatively minor changes, is the most important method in finding solutions to linear programming problems. As a first step, students are asked to find solutions to a system that contains more variables than equations. Normally when there are more variables than equations, there are an infinite number of solutions in a system of equations.
As a second step, students learn to bring a problem to the standard form and then to the tabular form. Students will learn to apply the simplex algorithm. Then, students use their newly acquired knowledge to solve two problems using the simplex algorithm.

Activities Plan:

Introductory activities (creation of interest, reference to real value issues, relation to background experiences etc)

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
2 weeks earlier than the classroom consideration	Each student receives a system with more variables than equations, which although it has an infinity of solutions, sometimes it is difficult to find a single solution: $\{x_2 + x_3 - 2x_5 = 7\ 4x_3 + x_4 + 3x_5 = 8\ x_1 - 2x$	Provide a document with written instructions
1 week earlier than the classroom consideration	Consider the conclusions reached by the students and move on to the next step.	Provide a document with written instructions

Development activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
15 min	**Standard form:** A linear programming problem is said to be in standard form when all constraints become equations and all non-negative variables. An inequality becomes equality by introducing a new variable, called the compensation variable, which represents the value left unused from the respective resource. **Tabular form:** A linear programming problem is in tabular form when all the constraints are equations with the right side greater than or equal to zero and when in each equation a variable appears that has the coefficient equal to +1 in a single equation and equal to 0 in all other equations. The variable with a coefficient equal to +1 in one equation and 0 in the others is called basic variable associated with the equation. All other variables that are not basic variables are called secondary variables. The value of the tabular form is that it provides an easy way to find a solution by setting the values of the secondary variables to zero and the values of the basic variables to the level corresponding to the right member of the equation. These solutions are called basic feasible solutions. These solutions correspond to the extreme points of the basic feasible region. We will illustrate the tabular form for the system received by the students: $\{x_2 + x_3 - 2x_5 = 7\ 4x_3 + x_4 + 3x_5 = 8\ x_1 - 2x_3 +$ The basic variables are x_2, x_4, x_1 and x_6. The basic feasible solutions are respectively $x_1 = 16;\ = 7;$ $x_3 = 0; x_4 = 8; x_5 = 0$ and $x_6 = 10$. A problem in tabular form can be written in a table. A table is a representation of the coefficients of the equations that satisfy the following conditions: 1. The coefficients of the objective function are written above the table of the coefficients of the equations. They represent the direct increase per unit of the objective function by increasing each variable by one unit, neglecting the effect of constraints. 2. To the left of the table are written the basic variables corresponding to each equation and its coefficient in the objective function. 3. At the base of the table are two lines:	Discussion on the issues Identify the important concepts **Determine:** Variables Objective functions Constrains on the variables

- the line z_j which represents the decrease in value of the objective function as a result of the increase by one unit of the variables x_j, decrease due to the effects of the constraints. It is obtained by multiplying the coefficients of each column by the objective function coefficient corresponding to the basic variable for that equation and summing them.

- The $c_j - z_j$ line represents the marginal profit line or the net valuation line, obtained by subtracting the zj values from cj for each column.

The base	c_j	x_1	x_2	x_3	x_4	x_5	x_6	
		3	3	10	1	5	1	
x_2	3	0	1	3	0	-2	0	7
x_4	1	0	0	4	1	3	0	8
x_1	3	1	0	-2	0	2	0	16
x_6	1	0	0	0	0	4	1	10
	z_j	3	3	7	1	7	1	87
	$c_j - z_j$	0	0	3	0	-2	0	

15 min

The **Simplex method** is an iterative procedure for solving linear programming problems brought to the tabular form. The simplex method generates new basic feasible solutions that increase the value of the objective function (or at least leaves it unchanged), by generating new tabular forms for the system of equations. When no improvement can be made, the optimal solution has been reached.

Mainly, the simplex method consists of 3 steps:

1. Finding the largest positive value for $c_j - z_j$. This will designate the pivot column. If there is no such value, then the optimal solution has already been found.

2. For each positive value in the pivot column there is the ratio: (right limb) / (the corresponding element in the pivot column). The minimum ratio sets the pivot line. At the intersection of the pivot column with the pivot line is the pivot element.

3. The new tabular form is generated as follows:

(a) Divide the pivot line by the pivot element;

(b) For all other lines, the new line generated in point (a) is multiplied. through the corresponding element in the pivot column and is extracted from the current line.

	(c) Fill the cells of the pellet and proceed to step 1. The procedure described above applies to maximization issues. For minimization problems we can proceed: (1) multiplying the objective function by -1 and maximizing, or (2) changing step 1 so that the most negative $c_j - z_j$ and stop are found if no such value has been found.	

Practicing Activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
In the classroom on the planned day for the lesson 20 min	Let the following be a linear programming problem: $$z = 2x_1 + 3x_2 + 4x_3 \rightarrow MAX$$ $\{x_1 + x_2 + x_3 \leq 30\ 2x_1 + x_2 + 3x_3 \geq 60\ x_1 - x_2 + 2x_3 = 20\ x_j \geq 0,$ a) Write the problem in standard form; b) Add the artificial variables needed to get the first tabular form and correct and match the objective function; c) Solve the problem by the simplex method; d) What changes should be made in order to minimize the objective function? ***Solution***: a) To bring the problem back to the standard form, the compensation variables s_1 and s_2 are introduced, so that the constraints change from inequality to equality. $$z = 2x_1 + 3x_2 + 4x_3 \rightarrow MAX$$ $\{x_1 + x_2 + x_3 + s_1 = 30\ 2x_1 + x_2 + 3x_3 - s_2 = 60\ x_1 - x_2 + 2x_3$ This form is not tabular because in the 2nd and 3rd constraint there are no variables with coefficient +1 in one and coefficient 0 in the others. b) To bring the problem to the tabular form, add the artificial variables a1 on the 2nd line and, respectively, a2 in the 3rd line. The objective function will also be modified by adding the new variables multiplied by the coefficient –M. Thus the tabular form will be:	**What are the important processes in a linear programming approach:** Deciding on what the variables are Determining the range of values of each variable Developing the equations/ inequalities connecting them (mathematical model) Determining the Objective function/s that is/ are to be optimised (maximised, minimized or otherwise)

		x_1	x_2	x_3	s_1	s_2	a_1	a_2	
The base	c_j	2	3	4	0	0	-M	-M	
s_1	0	1	1	1	1	0	0	0	30

a_1	-M	2	1	3	0	-1	1	0	60
$\leftarrow a_2$	-M	1	-1	2	0	0	0	1	20
	z_j	-3M	0	-5M	0	M	-M	-M	-80M
	$c_j - z_j$	3M+2	3	5M+4	0	-M	0	0	
				↑					

Because a_2 is an artificial variable and will become a secondary variable, we will abandon its column.

c) The 2nd iteration.

The base	c_j	x_1	x_2	x_3	s_1	s_2	a_1	
		2	3	4	0	0	-M	
s_1	0	$\frac{1}{2}$	$\frac{3}{2}$	0	1	0	0	20
$\leftarrow a_1$	-M	$\frac{1}{2}$	$\frac{5}{2}$	0	0	-1	1	60
x_3	4	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	0	20
	z_j	$-\frac{1}{2}M +$	$-\frac{5}{2}M -$	4	0	M	-M	-30M+40
	$c_j - z_j$	$\frac{1}{2}M + 2$	$\frac{5}{2}M + 5$	0	0	-M	0	
			↑					

The 3nd iteration.

The base	c_j	x_1	x_2	x_3	s_1	s_2	
		2	3	4	0	0	
$\leftarrow s_1$	0	$\frac{1}{5}$	0	0	1	$\frac{15}{4}$	11
x_2	3	$\frac{1}{5}$	1	0	0	$-\frac{2}{5}$	6
x_3	4	$\frac{3}{5}$	0	1	0	$\frac{-1}{5}$	13
	z_j	3	3	4	0	-2	
	$c_j - z_j$	-1	0	0	0	2	
						↑	

The 4nd iteration.

The base	c_j	x_1	x_2	x_3	s_1	s_2	
		2	3	4	0	0	
s_2	0	$\frac{1}{3}$	0	0	$\frac{5}{3}$	1	$\frac{55}{3}$
x_2	3	$\frac{1}{3}$	1	0	$\frac{2}{3}$	0	$\frac{40}{3}$
x_3	4	$\frac{2}{3}$	0	1	$\frac{1}{3}$	0	$\frac{50}{3}$
	z_j	3	3	4	$\frac{10}{3}$	0	$\frac{320}{3}$

	$c_j - z_j$ | -1 0 0 $\frac{-10}{3}$ 0 Thus the optimal solution is $x_1 = 0$, $x_2 = \frac{40}{3}$, $x_3 = \frac{50}{3}$, $s_1 = 0$, $s_2 = \frac{55}{3}$, and the optimal value is z = 320/3. d) The problem can turn into a minimization problem by changing the value of the coefficients of the artificial variables from –M to + M.	

Assessment activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
Next day 10 min	1. Solve the following problem using the simplex algorithm. Discuss the site's solutions. $$MAX\left[2x_1 + 3x_2 - x_3\right]$$ $\{3x_1 + 6x_2 \leq 30\ 4x_1 + 2x_2 + x_3 \leq 20\ x_2 + x_3 \leq 10\ x_j \geq 0,$ 2. It is considered that all the favorite food of children in a garden comes from one of the four "basic food groups" (chocolate, ice cream, mineral water and cake). At present, the following foods are available for consumption: gingerbread, chocolate ice cream, cola and pineapple cake. Each gingerbread cost 50 monetary units, each cup of chocolate ice cream 20 monetary units, each glass bottle 30 monetary units and each piece of cake and 80 monetary units. At least 500 calories, 6 units of chocolate, 10 units of sugar and 8 units of fat must be ingested every day. The nutritional content of each food unit is shown in the table below.	Discussion, Solutions

	calories	Chocolate (u.m.)	Sugar (u.m.)	Fats (u.m.)
gingerbread	400	3	2	2
ice cream	200	2	2	4
cola	150	0	4	1
cake	500	0	4	5

a) Identify the objective function;

b) Identify the system of constraints;

c) Solve the problem using the simplex algorithm.

Reflection and Closure

What are the basic assumptions that are leading to linear programming?

What are the advantages and what are the disadvantages of using the previous approaches?

Assignment for further work

Using the Internet, Study other methods and approaches for the consideration of linear programming.