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INNOMATH

Innovative enriching education processes for Mathematically Gifted Students in Europe

Reference Number: 2019-1-DE03-KA201- 059604

Network Damage

Topic: Graph, matrix

Target Group: Students at Grade 10 to 12 (13) (age range: 15-18 years old)

Goal/ Content/ Description:

Any large business or organization is a complex web of interconnected people and departments. An individual department functions at normal capacity when it receives resources and information that it needs from other departments. In turn, a department can provide resources and information to other departments when it is functioning normally. Therefore, the organization as a whole operates normally only when each individual department is functioning at normal capacity. What happens to an organization when one or more of its departments cannot operate at full capacity? How does this lost capacity, or damage, affect the rest of the organization?

The aim of this lesson is for students to develop digital skills. The aim of this lesson is to enable the students to understand simple matrix operations including multiplication, addition and transposes, also creating simple digraphs, all in the context of a certain decision analysis model.

Objectives:

General Mathematical Objectives

- To develop skills for problem solving
- To develop motives and positive affective tendencies for mathematics
- To identify/ develop/ create applications of the related concepts and processes in the real world
- To develop digital skills/ through the use/ exploitation of digital means as help/ support in calculations and representations
- To develop the ability to think algorithmically so that mathematical procedures can be transferred and translated into simple spreadsheets
- To develop skills for collecting and analyzing data and other information as they appear in the real world

Particular Mathematical Objectives

- Describe mathematical models for network damage
- Describe simple (calculation) commands of spreadsheets
- Transfer mathematical models and their calculations into spreadsheet commands

- Analysis of parameter changes and selection of optimal results

Materials/ Tools:

- Computer
- Spreadsheet incl. diagrams
- possible beamer
- possibly Power-Point presentations

Resources used by the teacher:

Introductory books on the use of spreadsheets
Articles, ppt presentations, YouTube videos on the subject of Excel by using the Internet
School Textbooks covering the topic
The materials in appendix
Further materials on the topic:

(a) http://www.algorytm.org/narzedzia/edytor-grafow.html

Resources for the student:

Articles, examples, exercises, ppt presentations, YouTube videos on the subject of Excel by using the Internet.
For this the teacher is to prepare a list of webpages in the mother language of the students.
School Textbooks covering the topic.
Work sheets prepared by the teacher (eg an example can be found in the appendix)

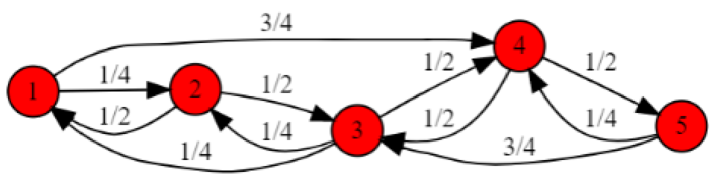
Approaches/ Methodology: giving students opportunities to solve problems well by understanding them well and creating links between different areas of mathematics and applying different skills. This teaching enables good learning to take place by treating students as thinking individuals who can operate mathematically.

Activities Plan:

Introductory activities (creation of interest, reference to real value issues, relation to background experiences etc)

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
one week before	organise a room with enough computers with spreadsheet software	
15 min	Present a problem: Network Damage (Appendix 1).	Discussion on the issues Identify the important concepts

Development activities

<table>
<tr><td>Time
When / length</td><td>Description of the activity</td><td>Instructions/ Hints/ Support/ Comments</td></tr>
<tr><td>15 min</td><td>Students draw a graph corresponding to a given matrix:
<table>
<tr><th></th><th>node 1</th><th>node 2</th><th>node 3</th><th>node 4</th><th>node 5</th></tr>
<tr><td>node 1</td><td>0</td><td>0,25</td><td>0</td><td>0,75</td><td>0</td></tr>
<tr><td>node 2</td><td>0,5</td><td>0</td><td>0,5</td><td>0</td><td>0</td></tr>
<tr><td>node 3</td><td>0,25</td><td>0,25</td><td>0</td><td>0,5</td><td>0</td></tr>
<tr><td>node 4</td><td>0</td><td>0</td><td>0,5</td><td>0</td><td>0,5</td></tr>
<tr><td>node 5</td><td>0</td><td>0</td><td>0,75</td><td>0,25</td><td>0</td></tr>
</table>
</td><td>If necessary, use the graph maker online (Appendix 3)</td></tr>
<tr><td>30 min</td><td>Students draw a graph corresponding to a given matrix


</td><td>A suggested solution can be found in the attached Excel spreadsheet (see used Excel formulas)</td></tr>
</table>

Practicing Activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
30 min	Students find values of state matrixes in successive time units: a) by multiplying the matrixes b) using spreadsheet	A suggested solution can be found in the attached Excel spreadsheet (Appendix 2).

Assessment activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments

15 min	Students suggest a way to repair a network.	Suggest giving the nodes (states) priorities. Consider different options. Perform a simulation in a spreadsheet.

Reflection and Closure

What influence on the network does damaging one node have? And what a few?
What are the advantages and disadvantages of using spreadsheets (digital perspective)?

Assignment for further work

The Repair Factor. When $p < 1$, then the evolution of the state vector is random and either damage spreads or repair occurs (but not both) at each time-step. R is the repair vector where:

$$R = [r_1, r_2, \ldots, r_n]^T$$

which is a 1 by n matrix consisting of repair units, r_j, assigned to node j. The repair vector allocates some fixed number of repair units, so

$$\sum_{j=1}^{n} r_j = R_{max}$$

Where R_{max} is the available quantity of repair units. At a time-step where repair occurs, the following formula calculates the new state vector:

$$X(t+1) = X(t) + R$$

where the repair vector must be defined so that no capacity value at a node exceeds 100.

The major question of importance is the best way to allocate the repair units to bring the network back to full capacity, or as a worst case scenario introduce stability to prevent the continued degradation of the system.

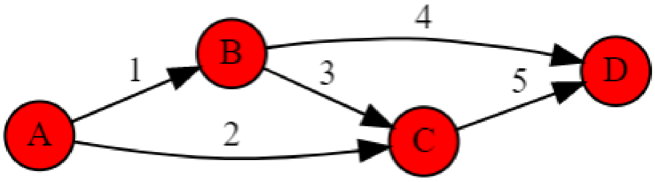
In an ideal situation, there would be enough repair units available to bring the system back to full capacity immediately. However, it is more realistic to limit the available repair units, R_{max}, for the model. When the number of repair units is a finite amount, repair must be allocated based on a different strategy. Some possibilities are:

1) Assign priority to the nodes.
    a. How do you assign priority?
2) Repair only the node where the initial damage occurred.

Appendix 1

When a company takes a quantitative risk analysis approach to disaster recovery planning, any number of mathematical models can be utilized to analyze the company. The model takes the organization as a complex structure of interconnected departments. The organization can implement this model in a simulation that is designed to identify vulnerabilities in the structure. The organization can use the results of the simulation to develop effective strategies to eliminate or protect these weaknesses in the system. Some possible models include the standard network flow model and the decision analysis model, which an organization can use to implement a quantitative approach to risk analysis.

Network models are used to simulate flow through oil pipelines, shipment of goods between factories and distribution points, telephone communications, and many other related activities. An example of a simple network is as follows:



A, B, C, and *D* are ***nodes***. The nodes are connected by ***arcs***, labeled 1 through 5. In this example, there are several paths from *A* to *D*, but all must pass through either *B* or *C* (or both). If the network flows model is compared to a road map, the nodes are intersections and the arcs are the roads that can be taken at that intersection.

In one application, the network models ***flow*** from a ***source*** (for example, node *A*) to a *sink* (node *D*). If there are capacities assigned to each arc and/or each node, then the model can be used to determine the maximum flow capacity for the entire network. If there are costs assigned to the flow along an arc, then the model can be used to find the cheapest way to transport materials (or information) from the source to the sink.

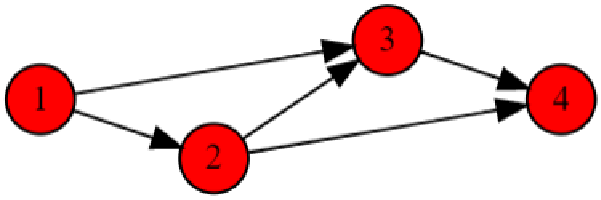
The standard network flow model presented above is a simple version of an actual network system. A simple model of this type is useful when there is information that needs to travel from one point to another. The model allows the user to look at the structure of the system in order to determine its flaws. For example, if the user knows the probability of failure for each path in a network, then the user may be able to change the connections in the network in order to improve the overall reliability of the system.

Sometimes the system is more complicated than the simple network flow model above. There is not always a source and a sink in a network model. Rather, the network model is set up where there are a number of possible choices at each node. The network flow starts at a source node and braches out based on the decisions made at that node. An arc connects this node to another node, where another decision must be made. In this type of network flow, there are many combinations of nodes and arcs because of the different combinations of decision that can be made at each node. This type of network flow model is also known as **a decision analysis model**.

The model of damage spread uses a network model for a complex organization. The nodes in the network correspond to the departments in the company. An arc connects two departments if there is some degree of (immediate) dependence between the two departments. There is a weight assigned to each arc as a measure of the strength of the dependence.

The mathematical model of damage spread assumes that one department (node) has lost some capacity. The state of the organization then evolves in discrete time-steps, either by letting damage spread from each damaged node to the adjacent nodes or by applying repair to the damaged nodes. This section describes the model in more detail and includes some simple examples to illustrate the basic properties of the damage spread process as well as the ways that repairs strategies can limit the spread of damage and return the organization to operating conditions.

The structure of a network or directed graph is recorded in an ***adjacency matrix***. In a standard adjacency matrix, an entry A_{ij} = 1 if node *j* has an incoming arc from node *I* and $A_{ij} = 0$ otherwise. For example, given the network:



The adjacency matrix would be:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Each row of the adjacency matrix represents a different node. For example, the first row represents node 1 and indicates that there are arcs from node 1 to nodes 2 and 3. Note that this is a one way connection. There is an arc from node 1 to node 2, but there is not a flow from node 2 to node 1 (the entry A_{21} = 0.).

In the ***weighted adjacency matrix***, any non-zero entry indicates a level of dependence of one department on another. A small value for A_{ij} indicates that lost capacity at node *i* will have a small impact on the working capacity at node *j*. If, on the other hand, A_{ij} is close to 1, then lost capacity at node *i* translates almost completely into the same loss in capacity at node *j*. The weighted adjacency matrix describes how damage at each node will change at each time-step.

The underlying assumption is that once an initial node is damage, the damage can spread to the neighboring nodes. Nodes that have an incoming connection from the damaged node feel the effects of the damage at the first time-step. The idea is that each node requires some resources in order to function and these resources are provided by other nodes in the organization. For example, assume that there are only two nodes and suppose that node 2 has a connection from node 1 with weight 0.50. If node 1 incurs damage or loss of resources of 5% at the initial time step, its capacity is reduced to 95%. At the next time-step, 50% of the damage at node 1 spreads to node 2 due to the weight of 0.50. As a result, node 2's capacity is reduced by 2.5% to 97.5%.

At each step, it is necessary to take the current state and compute the state of the next time step. The model assumes that *either* damage spreads *or* repair occurs. The current state of the organization is recorded in a vector *X*:

$$X(t) = \left[X_1(t), X_2(t), \ldots X_n(t)\right]^T$$

is an *n* by 1 matrix, where *n* is the number of nodes in the network, whose entries record the current capacity for each department. For example, in a three-node network at time 4, the state vector *X* may be

$$X(4) = \begin{bmatrix} 50 & 85 & 0 \end{bmatrix}^T$$

This vector indicates that, at time 4, $X_1(4) = 50$, which means node 1 is at a capacity of 50%, capacity at node 2 is 85%, and node 3 has failed and is at 0%. The *damage level* at a certain node is just 100 minus the current state value.

A difference equation used to describe the evolution of the state is given by:

$$X(t+1) = \left[X(t) - (1-p) \cdot A^T \cdot (100 - X(t)) + p \cdot R\right]^+$$

where:

A is the weight adjacency matrix

p is the probability of that repair occurs at a time step

100 is an *n* x 1 column vector of 100's

R is the *repair vector*, which will be a function of the current state: *R = R(X(t))*

The superscript + indicates that only the positive part of the argument is returned, so that a node cannot further degrade when it has reached 0.

Appendix 2

file: spreadsheet-1-matrix-graph.xlsx

Appendix 3

Graph maker: http://www.algorytm.org/narzedzia/edytor-grafow.html