

The following example is a guiding format for developing a Lesson Plan in a situation of supporting the students working in the context of the INNOMATH project. This mathematical content is expected to be useful for the students in their effort for solving industrial problems.

Lesson/ Learning Plan

Topic: Specify a particular mathematical topic

Introduction to Linear Programming: Simplex method

Target Group: Specify age, grade level and mathematical background of the students

“Gifted” Students in a high school at grade level 11 or 12 (5th or 6th year in a secondary school)

Mathematical background of the students:

Basic Concepts of Matrices,

Functions, Graphs of linear functions,

Solving linear equations and systems of equations,

Solving linear inequalities. Representing the solutions on graph diagrams

Capability of using graph software and interpreting the concepts involved

Capability of using a spreadsheet software

Goal/ Content/ Description: (Provide a brief description of the content of the lesson as well as the general goals in relation to the prospects of usefulness, applications and mathematical value of the particular mathematical topic.)

in the introduction of linear programming, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

Objectives: Will be able to solve some industrial problems that involves more than two constraints that can be solved using the graphical method.

General Mathematical Objectives

To develop skills for problem solving

To develop motives and positive affective tendencies for mathematics

To identify/ develop/ create applications of the related concepts and processes in the real world.

To develop mathematical skills/ through the use/ exploitation of mathematical topics or means as help/ support in modelling, calculations and representations,

To develop digital skills/ through the use/ exploitation of digital means as help/ support in calculations and representations,

To develop skills for collecting and analyzing data and other information as they appear in the real world

To exploit the flipped classroom method for supporting the various processes.

Particular Mathematical Objectives

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

Materials/ Tools: for example ppt, use of graphing software etc.

Computers and or scientific calculators.

Graphing software (eg Geogebra, Graph etc)

Power Point Presentations

Graph Paper

Simplex method application

Resources used by the teacher:

Introductory books on linear programming

Articles, examples, exercises, ppt presentations, YouTube videos (eg Khan Academy) on the topic by using the Internet. Appendix.

Resources for the student:

Articles, examples, exercises, ppt presentations, YouTube videos (eg Khan Academy) on the topic by using the Internet. For this the teacher is to prepare a list of webpages in the mother language of the students. Work sheets prepared by the teacher. The teacher can use the following example to provide a sample of the procedure and a step-by-step guidance.

<https://www.youtube.com/watch?v=XK26I9eoSI8>

Approaches/ Methodology: Describe briefly the approaches to be used (e.g. problem solving project, flipped classroom,... ,the role of the teacher, etc)

The flipped classroom approach will be used in order to give to the student the possibility for investigation, and access of information, and watching videos demonstrating the approach and posing the problem of linear programming.

The project based problem solving approach will be utilised in order to help the students to acquire the skill to investigate the prospect of extending the consideration of linear programming for non-graphical methods (in the case of more than two variables)

Activities Plan:

Introductory activities (creation of interest, reference to real value issues, relation to background experiences etc)

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
Previous Day	Revise the concepts of solving linear equations and determining regions that are satisfied by linear inequalities	See Appendix 1 for a problem
10 min	Present a problem that is faced by the real world eg. A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair, \$30 for a table, and \$25 for a bookcase, how many units of each should be manufactured to maximize profit?	Discussion on the issues when more variables are involved in the problem and the difficulty to solve the problem using the graphical approach.

Development activities

Time	Description of the activity	Instructions/ Hints/ Support/ Comments
15 min	Provide the Simplex method 1. Set up the problem. That is, write the objective function and the inequality constraints. 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality. 3. Construct the initial simplex tableau. Write the objective function as the bottom row. 4. The most negative entry in the bottom row identifies the pivot column. 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element. The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored. 6. Perform pivoting to make all other entries in this column zero. This is done the same way as we did with the Gauss-Jordan method. 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4. 8. Read off your answers. Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.	What are the important processes in a Simplex method : 1. Identify and set up a linear program in standard maximization form 2. Convert inequality constraints to equations using slack variables 3. Set up the initial simplex tableau using the objective function and slack equations 4. Find the optimal simplex tableau by performing pivoting operations. 5. Identify the optimal solution from the optimal simplex tableau.

Practicing Activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
40 min	<p>Implement the plan by using a worked example on a simpler problem:</p> <p>Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes \$40 an hour at Job I, and \$30 an hour at Job II, how many hours should she work per week at each job to maximize her income?</p>	Discussion, Solutions
10 min	Suggest the consideration of problems with more than 3 variables and suggest to work at home referring to supporting webpages.	
10 min	<p>Introduce the simplex method algorithm.</p> <p>Students can watch a youtube link on how to use the programme and experience some programming techniques which will be an extension of the topic.</p>	

Reflection and Closure

For maximization we set the inequalities in which direction?

What are the slack variables?

How do we find the pivot column?

How do we find the pivot row?

Assignment for further work

Using the Internet, Study other methods and approaches for the consideration of quadratic approximation.

What is the Jacobi Method?

What is the Hessian Matrix?

Appendix

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes \$40 an hour at Job I, and \$30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated as:

Maximize subject to: $z = 40x_1 + 30x_2$

$$x_1 + x_2 \leq 12$$

$$2x_1 + x_2 \leq 16$$

$$x_1 \geq 0; x_2 \geq 0$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1 , and we get $x_1 + x_2 + y_1 = 12$.

Here the variable y_1 picks up the slack. In this problem, if Niki works fewer than 12 hours, say 10, then $y_1 = 2$. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We write the objective function $z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + z = 0$.

After adding the slack variables, our problem reads

Objective function $-40x_1 - 30x_2 + z = 0$

Subject to constraints: $x_1 + x_2 + y_1 = 12$

$$2x_1 + x_2 + y_2 = 16$$

$$x_1 \geq 0; x_2 \geq 0$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12 y_1
2	1	0	1	0	16 y_2
-40	-30	0	0	1	0 Z

↑

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $z = 40x_1 + 30x_2$, it will make more sense to increase the value of x_1 , rather than x_2 . The variable x_1 , represents the number of hours per week Niki works at Job I. Since Job I pays \$40 per hour as opposed to Job II which pays only \$30, the variable x_1 , will increase the objective function by \$40 for a unit of increase in the variable x_1 .

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x_1	x_2	y_1	y_2	Z			
1	1	1	0	0	12	y_1	$12 \div 1 = 12$
2	1	0	1	0	16	y_2	$\leftarrow 16 \div 2 = 8$
-40	-30	0	0	1	0	Z	

↑

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1 . But we cannot choose any value for x_1 . Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again, the answer is no because the preparation time for Job 1 is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is $16 \div 2 = 8$.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

Question Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable**, and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1 , and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.

$$\begin{array}{ccccc|c}
 x_1 & x_2 & y_1 & y_2 & Z & \\
 1 & 1 & 1 & 0 & 0 & 12 \\
 \boxed{1} & 1/2 & 0 & 1/2 & 0 & 8 \\
 \hline
 -40 & -30 & 0 & 0 & 1 & 0
 \end{array}$$

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1. We get

$$\begin{array}{ccccc|c}
 x_1 & x_2 & y_1 & y_2 & Z & \\
 0 & 1/2 & 1 & -1/2 & 0 & 4 \\
 \boxed{1} & 1/2 & 0 & 1/2 & 0 & 8 \\
 \hline
 -40 & -30 & 0 & 0 & 1 & 0
 \end{array}$$

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

$$\begin{array}{ccccc|cc}
 x_1 & x_2 & y_1 & y_2 & Z & & \\
 0 & 1/2 & 1 & -1/2 & 0 & 4 & y_1 \\
 \boxed{1} & 1/2 & 0 & 1/2 & 0 & 8 & x_1 \\
 \hline
 0 & -10 & 0 & 20 & 1 & 320 & Z
 \end{array}$$

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$ we obtain $x_1 = 8$, $y_1 = 4$ and $z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\left[\begin{array}{ccc|c}
 x_1 & y_1 & Z & C \\
 0 & 1 & 0 & 4 \\
 1 & 0 & 0 & 8 \\
 0 & 0 & 1 & 320
 \end{array} \right]$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$ and $z = 320$. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be \$320.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x1	x2	y1	y2	Z			
0	1/2	1	-1/2	0	4	y1	$\leftarrow 4 \div 1/2 = 8$
1	1/2	0	1/2	0	8	x1	$8 \div 1/2 = 16$
0	-10	0	20	1	320	Z	

↑

We make the pivot element 1 by multiplying row 1 by 2, and we get

x1	x2	y1	y2	Z		
0	1	2	-1	0	8	
1	1/2	0	1/2	0	8	
0	-10	0	20	1	320	

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x1	x2	y1	y2	Z		
0	1	2	-1	0	8	x2
1	0	-1	1	0	4	x1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or}$$

$$z = 400 - 20y_1 - 10y_2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled y_1 and y_2 are not such columns, we arbitrarily choose $y_1 = 0$, and $y_2 = 0$, and we get

$$\left[\begin{array}{ccc|c} x_1 & x_2 & Z & C \\ 0 & 1 & 0 & 8 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 400 \end{array} \right]$$

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z=400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to \$400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

Link with further examples on YouTube:

[\(6\) !\[\]\(0f848bbd71cef6b345273b16f905912a_img.jpg\) How to use the simplex method to solve maximization problems \(Question 1, Easy\) - YouTube](#)