Problem Solving Methods

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Problem solving in mathematics classes

Literature:

G. Polya, *Schule des Denkens*, Tübingen, Basel. Francke, 1949 (4.Aufl. 1995).

A. Posamentier, W. Schulz (Hrsg.): *The Art of Problem Solving*, 1996, S. VIII.

H. Schwarz, *Heuristische Strategien des Problemlösens. Eine fachmethodische Systematik für die Mathematik.* Münster, WTM, 2006.

Difficulties with problem solving in mathematics classes

Problem solving = Solve task without known routine procedures

- No approach
- Missing strategy
- $\rightsquigarrow$ "Idle mode" $\rightsquigarrow$ Disciplin problems

Problem avoidance strategy for teachers:
Standard tasks

PISA

In an international comparison, German students perform relatively well in technical tasks. Their weakness lies in modeling demanding inner-mathematical contexts. This performance profile is certainly related to the calculus orientation of German mathematics instruction, as shown in earlier studies.

"Die deutschen Schülerinnen und Schüler schneiden im internationalen Vergleich bei technischen Aufgaben relativ gut ab; ihre Schwäche liegt in der Modellierung anspruchsvoller innermathematischer Kontexte. Dieses Leistungsprofil hängt sicherlich mit der in früheren Studien aufgezeigten Kalkülorientierung des deutschen Mathematikunterrichts zusammen."

Deutsches PISA-Konsortium (Hrsg.): *PISA 2000*, Leske + Budrich, Opladen 2001, S. 178.

Definition
Task categories

Definition, Task categories

Problems (or more generally tasks) are mental demands on individuals, where an

Initial state	A	by means of a
Transformation	T	is changed into a
Target state	Z	.

- Example task: everything is known
  (A;T;Z) $\rightarrow$ (k;k;k)
- Problem: T is unknown (non routine available) $\rightsquigarrow$ Overcome barrier
  (A;T;Z) $\rightarrow$ (k;u;k)
- Open Problem: A or Z unknown, Several possibilities for T

Boundaries between categories not sharp
Classification depends on the student

Problem: Definition

Problems from a psychological point of view

Problems are characterized by the fact that a given state is experienced as being unsatisfactory and therefore needs to be changed. However, the necessary measures or steps cannot simply be recalled from memory, but have to be constructed (otherwise it is a task).

"Probleme sind dadurch gekennzeichnet, dass ein gegebener Zustand als unbefriedigend erlebt wird und deshalb verändert werden soll. Jedoch können die dazu erforderlichen Maßnahmen oder Schritte nicht einfach aus dem Gedächtnis abgerufen werden, sondern müssen konstruiert werden (sonst handelt es sich um eine Aufgabe)."

STROHSCHNEIDER, S.: *Problemlösen in Deutschland*. In: *Denken in Deutschland*. Hans Huber: Bern, Göttingen, Toronto, Seattle 1996, S. 18.

Barriers in problem solving

- Objective barriers: Necessary mathematical contents not known
- Subjective barriers: Uncertainty which knowledge is suitable
- Removal of all barriers $\rightsquigarrow$ Routine task
- Barriers to high $\rightsquigarrow$ Failure
- Do not remove barriers but “Lower the bar”

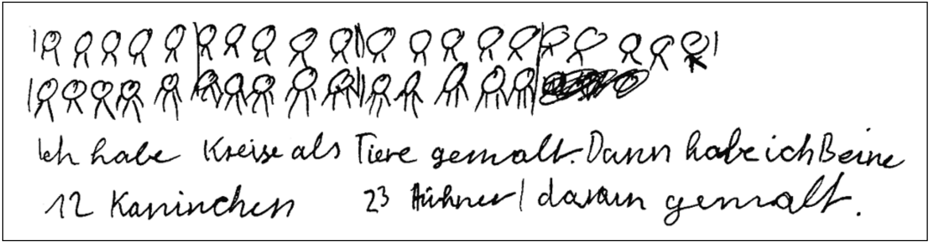
Examples from elementary school

Chicken and rabits

M. Grassmann, Mathetreff (Students of 4th grade)

Linear systems of equations – Chicken and rabits

In one hutch are rabbits and chicken. The animals have 35 heads and 94 legs. How many rabbits and how many chicken are there?

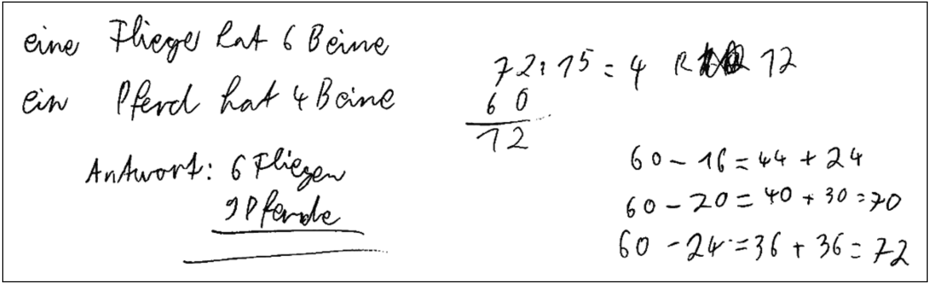


“I drew circles as animals. Then I added legs.”

Horses and Flies – Solution 1

Horses and flies

One day, 15 animals are counted in a stable – horses and flies. Together they have 72 legs. How many horses and how many flies are there?


eine Fliege hat 6 Beine
ein Pferd hat 4 Beine

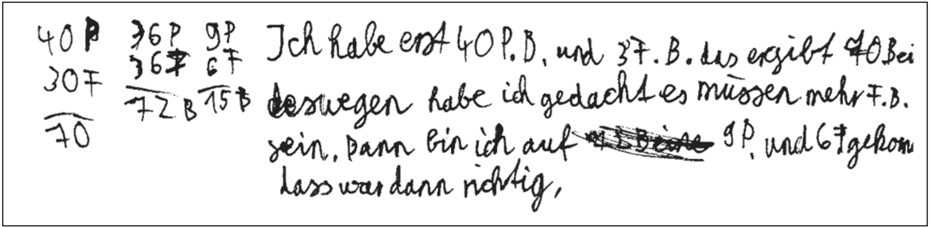
Antwort: 6 Fliegen
9 Pferde

72 : 15 = 4 R 12
60
72

60 − 16 = 44 + 24
60 − 20 = 40 + 30 = 70
60 − 24 = 36 + 36 = 72

Horses and Flies – Solution 2

Jon:


40 P 30 F 70

36 P 36 F 72 B

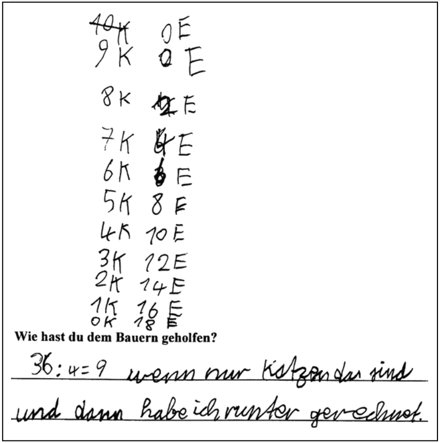
9 P 6 F 15 B

Ich habe erst 40 P. B. und 37 F. B. das ergibt 70 Bei deswegen habe ich gedacht es müssen mehr F. B. sein. Dann bin ich auf 9 P. und 6 F gekom dass war dann richtig.

Linear Diophantine Equations

Cats and ducks

On a farm live cats and ducks. Max has counted 36 legs. How many cats and how many ducks can there be?



Realization:
several possibilities
How to find all?

Cats and Ducks: another Solution

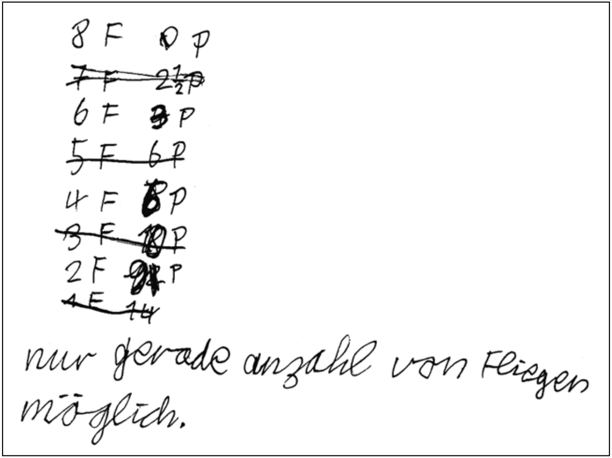
Leo: “A cat can always be replaced by two ducks.”

> Die Enten müssen immer gerade Zahlen bilden, weil 2 Enten 1 Katze sind. Wenn die Entenzahl ungerade ist, ergibt es keine volle Katze.

Horses and flies – II

Horses and flies (II)

The farmer has horses and flies in his stable. He has counted 48 legs. How many horses and how many flies can there be?



Content oriented or
concrete-experimental
solving of problems

- Calculation problems: use table, drawing, measurement
- "First solution" often only approximation $\rightsquigarrow$ Order of magnitude, assumption

Further examples
Grades 5 to 10

Find possible solutions that students might give.
Work in small groups.

Some examples for problem tasks

Frequent references to the domains

- Geometry
- Number theory
- Combinatorics
- Elementary logic

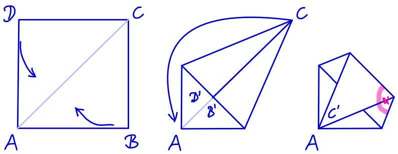
Little basic knowledge – Willingness to puzzle and mentally rearrange is encouraged.

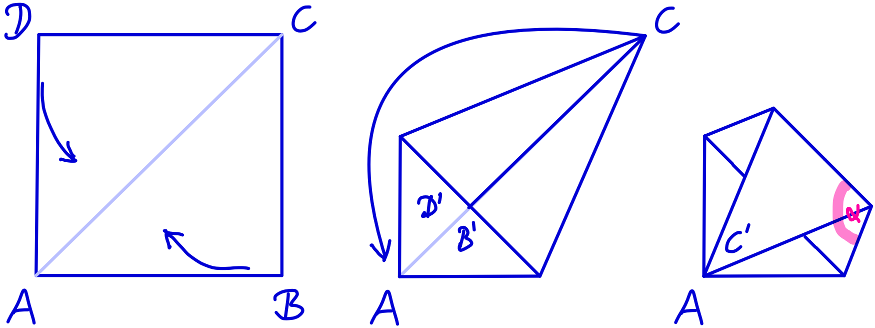
Example: Grade 7/8 ($*$)

A square paper is folded into a pentagon as shown in the figure:

1. The sides BC and CD are folded on the diagonal AC.
2. C is folded onto A.

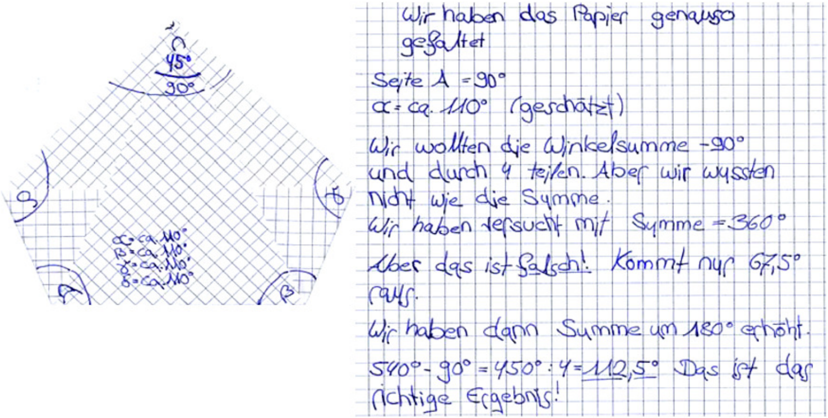
Find the angle a without the help of a protractor.
Can you find as many other angles as possible without measuring?





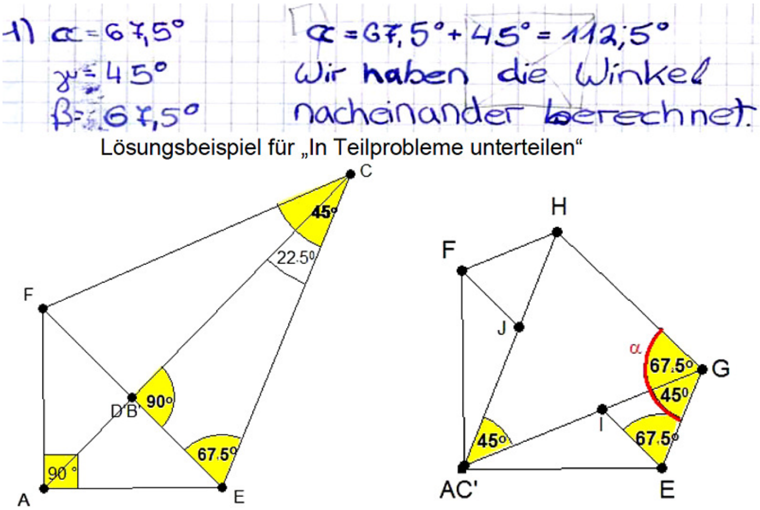
Winkel α: 112°
Erklärung: Wir haben es nachgefaltet und dann gemessen. Aber wir haben das Nachgefaltete auf ein Blatt geklebt und den Winkel verlängert.

Soved by folding and measuring



First: estimate

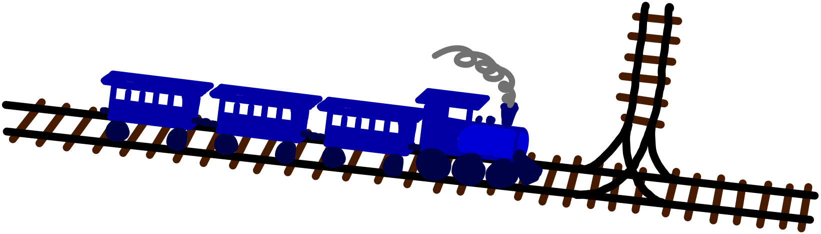
Then: assume the four angles, which are not right angles,
have the same size and guess the interior angle sum of a pentagon



Example of dividing into subproblems

Rearranging and Sorting

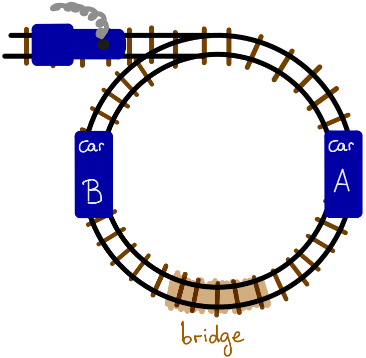
1. A train is to turn around on a single track line. There is only one very short siding that can accommodate at most the locomotive or one car. Give a method that can be used to turn the entire train around.



Rearranging and Sorting

In a marshalling yard, a locomotive is approaching a circular piece of track on which two empty wagons are standing. Between them is a bridge strong enough to support a car, but not the locomotive. The locomotive driver is supposed to switch the two wagons. Give a sequence of shunting steps that solves this task.

Thereby each car and the locomotive can be coupled and uncoupled at both sides. However, all coupling maneuvers are performed only when the train is stationary, shunting approaches are not allowed. The bridge is just as long as a wagon, and the locomotive must leave the circular track at the end.



Thinking and brainteasers, logic

Three red candies and three green candies are distributed in three boxes, where each box contains candies. The boxes are marked with “GG”, “GR” and “RR”. However, in no case does the inscription correspond to the contents of the box.
One candy may be taken from one box with the eyes closed. The color of the candy can be determined only after closing the box. From which box do you have to take a piece of candy from to be able to tell the exact contents of all the other boxes?

Thinking and brainteasers, logic

Mysterious family: A couple had less than 10 children. They are boys and girls. Each girl has as many sisters as brothers. However, each of the boys has only half as many brothers as sisters. How many are there exactly?

Thinking and brainteasers, logic (∗)

Klaus tells about his family. “I am more than 10 years old, but not yet 20 years old. My mother is one year older than my father and 12 times as old as my sister. My father is 21 years older than me.” How old are each of the family members?

Thinking and brainteasers, logic

Mike, Thomas, Reiko and Alfons were playing soccer in the yard and broke a window. When the incident was investigated, the boys made the following statements:

- Mike: "The window was broken by Thomas or Reiko."
- Alfons: "Reiko did it."
- Thomas: "I didn't break the window."
- Reiko: "Neither did I."

Their teacher, who knew the boys well, said, "Three of them speak always speak the truth."
Who broke the window?

Combinatorics tasks

Anna, Barbara, Cecilie, Doris and Erika have a race on the sports field. There are 5 lanes next to each other.

a) How many ways are there to distribute the 5 girls on the running tracks?

b) Barbara and Erika are friends and want to run next to each other. How many possibilities are there, if the wish of both of them is considered?

Combinatorics tasks ($*$)

From the set of letters $\{\texttt{p}, \texttt{r}, \texttt{o}, \texttt{d}, \texttt{u}, \texttt{c}, \texttt{t}\}$ should be formed words (also meaningless) with 4 letters. What is the number of possible word formations, if

a) No letter may be repeated?

b) Repetitions are allowed?

c) A word should consist of 2 consonants and 2 vowels of the given set of letters and repetitions are not allowed?

Tasks for elementary number theory

In the four-digit secret number on Mr. Muller's check card the first and fourth digits match. The two middle digits are also the same. The number that is formed from the last two digits is by 2 larger than the sum of the digits of the secret number. Justify that there can be only one secret number with these properties and determine it.

Tasks for elementary number theory

A gambler counted through a sum of ducats won, which was less than than 400. If he counts them in twos, threes, fives, and sevens, there will be one left over. If he counts them by twelves, he has seven ducats left over. How many ducats has he won?

Tasks for elementary number theory

Determine all numbers with the following properties:

(a) The number has three digits and contains three different digits, which are all prime numbers,

(b) The number is additionally divisible by each of the numbers denoted by its digits.

Number Puzzle

- Which letter stands for which digit?

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

Tetrahedron

Given a tetrahedron whose faces are congruent triangles and a point P inside. Prove, that the sum of the distances of P to all faces of the tetrahedron is constant.

A great tool:
Polya-Questions

Polya-Questions

Polya: *Schule des Denkens*, inner book cover

1. Understanding the task

- What is unknown? What is given? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or overdetermined? Or contradictory?
- Draw a figure! Insert a suitable name!
- Separate the different parts of the condition! Can you write them down?

Polya-Questions

2. Coming up with a plan

- Have you seen the task before?
- Do you know a related task? Do you know a theorem, that might be necessary?
- Consider the unknown! And try to think about a task that you know and that has a similar unknown.
- Here is a task that is related to yours and is already solved. Can you use it? Can you use its result? Can you use its method?
- Can you express the task differently? . . . Go back to the definition!

Polya-Questions

2. Coming up with a plan

- Can you solve part of the problem?
  - Keep only one part of the conditions and drop the other. How far is the unknown then determined, how can I change it?
  - Can you think of any other data that would be suitable to determine the unknown?
  - Can you change the unknown or the data or, if necessary, both, so that the new unknown and the new data are closer to each other?
- Did you use all the data? Did you use the whole condition? Did you take into account all the essential terms contained in the task?
- Geometry: Can you draw helping lines? Could a model help? Can you find congruent triangles?

Polya-Questions

3. Executing the plan

- When you carry out your plan of solution, check each step. Can you clearly see that the step is correct?

Polya-Questions

4. Review

- Can you check the result? Can you verify the proof?
- Can you derive the result in different ways? Can you see it at first sight?
- Can you use the result or method for any other task?

Cognitive structures
as a prerequisite for
problem solving

Cognitive structures as a prerequisite for problem solving

Cognitive structures consist of two parts, the epistemic and the heuristic structure.

Epistemic structure (episteme = Knowledge)

- irregular three-dimensional mesh
  - Nodes: Terms
  - Connections: Relations between terms

Heuristic structure (Heurismen = "Finding method")

- Heuristic structure: totality of all heurisms.
- Heurism: Structure that organizes the individual operators within a problem-solving process and controls the individual operators
- Epistemic structure like net – heurism like "octopus" reordering nodes, disconnecting links, forming links

Heuristic Strategies

General Heuristic Strategies

heurískein (greek): find, discover

Heuristik:

- Doctrine or science of the methods of solving problems (Duden)
- Doctrine of the methods and rules of discovery and invention (Polya)
- Teaching the procedures of finding true statements (textbook Fokus 8)

Sometimes instead of heuristic strategies also heuristic principles or "heurisms"

General Heuristic Strategies

Overview of heuristic strategies[1]

1. Working backwards
2. Find a pattern
3. Take another point of view
4. Solve a simpler analogous problem
5. Consider extreme cases
6. Attempt a visualization
7. Intelligent guessing and checking
8. Find necessary and sufficient conditions
9. Try to set up a sequence
10. Specialize without loss of generality

[1] A. Posamentier, W. Schulz (Hrsg.): *The Art of Problem Solving*, 1996, S. VIII.

General Heuristic Strategies

Overview of heuristic strategies[2]

11. Systematic and complete case distinction
12. Use a computer
13. Try to make a conclusion
14. Organize the data
15. Looking for an approximation of the solution
16. Determine characteristic properties of objects
17. Specialize
18. Generalize

[2] A. Posamentier, W. Schulz (Hrsg.): *The Art of Problem Solving*, 1996, S. VIII.

Heurisms of variation of representation

[3]

System change between everyday language and formal language with iconic elements

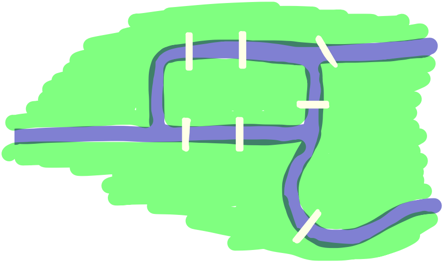
- Changing forms of representation (enactive, iconic, symbolic)
- Switch between linguistic formulations

[3] Classification of heurisms:
H. Schwarz, *Heuristische Strategien des Problemlösens. Eine fachmethodische Systematik für die Mathematik.* Münster, WTM, 2006.

Bridge Problem of Königsberg

Euler, 1736

Below is a map of Königsberg at the beginning of the 18th century. The two arms of the Pregel River, over which there are seven bridges in total, flow around an island (the "Kneiphof"). Is it possible to make a tour of the former Königsberg, using each bridge exactly once?



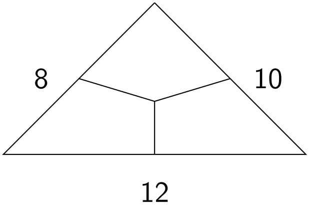
Who-is-who with pets

A family lives with their pets Tramp, Tiger, Carlo and Maxi under one roof, in peaceful community of human, dog, cat, hamster and parrot. Maxi is smaller than Tiger, who in turn is bigger than the dog. Carlo is older than the hamster, who gets along better with the parrot than with Maxi. Surprisingly, the hamster and the parrot are not afraid of Tiger. Find out the names of each animal.

Heurisms of variation of representation

Further options:

- Translating algebraic problems to geometric problems (or vice versa),
- Formulation of mathematical problems in the language of linear algebra,



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Heuristic
Strategies
More Examples

Heurisms of variation of the problem

Reformulation and Analogy

- Prominence in problem solving:
  Variations of the problem and consideration of analogous problems
- Many "Polya Questions" aim at this:
  - Have you seen this task before? Or have you seen the same task in a slightly different form?
  - Do you know of a related task?
  - Can you express the task differently? Can you express it in different ways?

Example (Analogy)

- Elevators: An elevator goes from the groud floor to the floors 1-20. There are 20 persons in the elevator who are independent of each other. Each with equal probability to leave on one of the floors. New persons do not get on during the ride. What is the expected value for the number of floors on which the elevator stops?
- Duck Hunting: Ten hunters, all perfect shooters (every target is hit), lie in wait for ducks in front of a rock. Soon ten ducks land. The hunters can only shoot once, and they can't make out who is shooting at which duck. Therefore, they shoot at the same time, but each chooses his victim at random.
    - What is the probability of survival of the seventh duck (or the eighth duck or . . . or the nth duck)?
    - How many ducks survive on average if this experiment is repeated many times?

Further heurisms of variation of the problem. . .

- Variation of perception by reorganization,
- Invariance principle and Symmetry principle,
- Generalization, Specialization, Principle of extremes (consideration of extreme cases)
- . . .

Example of the Invariance principle

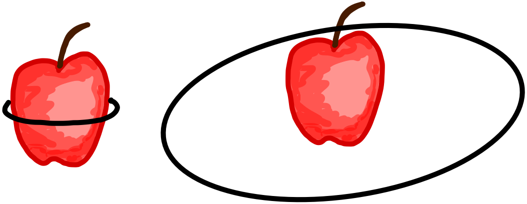
We consider a chessboard. We call a change valid, if all squares of a row (or column) change their color.

Is it possible to make an arbitrary sequence of valid changes so that a single black field remains?

Example (Generalizing)

Rope around the globe

A rope is placed around an apple ($d \approx 6cm$), lengthened by a meter and tightened in such a way that it again forms a circle, which has the same distance to the apple at every point.
Can a mouse then slip under the rope?



In the same way, a rope is laid along the equator around the globe ($d \approx 12.700\ \mathrm{km}$), lengthened by one meter and tightened so that it again forms a circle, which has the same distance to the equator at every point.
Can a mouse then slip under the rope?

Heurisms of Induction

Conclusions from the particular (or from a set of particulars) to the general are among the most important methods of gaining knowledge.

- Securing knowledge will not always succeed

$\rightsquigarrow$ Also heurisms of unfinished induction[4]
(in contrast to complete induction or mathematical induction)

[4] inductio (lat.): the leading into

Polya, 1962

Mathematics is considered a demonstrative science. But this is only one of its aspects. The finished mathematics, presented in finished form, appears to be as purely demonstrative. It consists only of proofs. But the mathematics in the making is like any other kind of human knowledge that is in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of a proof before you work out the details. You have to combine observations and follow analogies; you have to try again and again. The result of the mathematicians creative activity is demonstrative reasoning, is a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics is to reflect to some extent its invention, it must have a place for guessing, for plausible reasoning.

"Die Mathematik wird als demonstrative Wissenschaft angesehen. Doch ist das nur einer ihrer Aspekte. Die fertige Mathematik, in fertiger Form dargestellt, erscheint als rein demonstrativ. Sie besteht nur aus Beweisen. Aber die im Entstehen begriffene Mathematik gleicht jeder anderen Art menschlichen Wissens, das im Entstehen ist. Man muß einen mathematischen Satz erraten, ehe man ihn beweist; man muß die Idee eines Beweises erraten, ehe man die Details ausführt. Man muß Beobachtungen kombinieren und Analogien verfolgen; man muß immer und immer wieder probieren. Das Resultat der schöpferischen Tätigkeit des Mathematikers ist demonstratives Schließen, ist ein Beweis; aber entdeckt wird der Beweis durch plausibles Schließen, durch Erraten. Wenn das Erlernen der Mathematik einigermaßen ihre Erfindung widerspiegeln soll, so muß es einen Platz für Erraten, für plausibles Schließen haben." Polya, 1962 [5]

[5] Polya, G.: *Mathematik und plausibles Schließen*. Bd. 1. Basel: Birkhäuser, 1962.

Heurisms of Induction

- Inductive reasoning: reasoning from particulars to generalities (laws).
- In particular: (systematic) trial and error, searching for patterns and laws, setting up of sequences

Unfinished inductive conclusions can also lead to misconceptions

Example: Treasure hunt

Jim has arrived on Treasure Island with Billy Bones' treasure map and wants to recover the treasure.

> Walk from the gallows to the palm tree, counting your steps. Turn right by 90° and walk as many steps as you walked from the gallows to the palm tree. Put a stick in the ground.
> Go back to the gallows and walk to the well, again counting the steps. Turn left by 90 degrees and walk as many steps as you walked from the gallows to the well. Put a stick in the ground.
> The treasure is exactly in the middle between the two sticks.

Jim finds the palm tree and the well without any problems. However, there is no trace of the gallows. Can he still find the treasure?

The treasure island

Heurisms of reduction

Some of the heurisms of reduction are related to the solution of quite specific problems.

- Heurism "La Descente Infinie – the infinite Descent" – very well suited for incommensurability proofs.[6]
- Modularization:
  Decompose problems into subproblems (or use existing "modules" for more complex problems)
  - Computer Science
  - Geometric constructions

[6] siehe z.B. H. Schwarz, *Heuristische Strategien des Problemlösens*, S. 200ff.

Heurisms of reduction

Working backwards

"*We want to start from what is demanded and assume that what is sought is already found.*" POLYA).

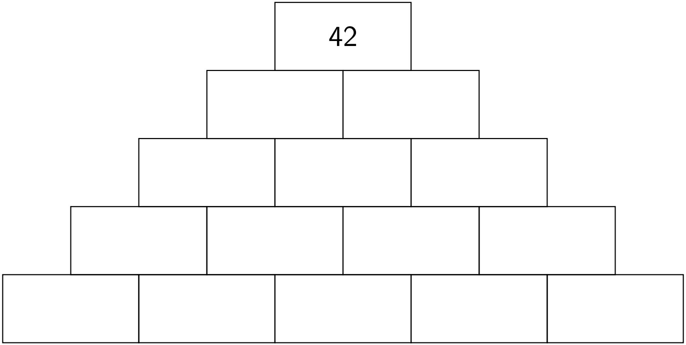
- What do you need to get the final result?
- Step by step to the initial state

⇝ Discover correct sequence of operations in reverse order

Example Working backwards (∗)

A man goes apple picking. To get to the city, he has to pass through 7 gates. At each gate there is a guard who asks for half of his apple and one more apple. At the end, the man has only one apple left. How many apples did he had at the beginning?

Example Working backwards: Wall of numbers ($*$)



What is the smallest sum of the numbers in the lowest row if only natural numbers (> 0 resp. ≥ 0) are allowed?

Example Working backwards (∗)

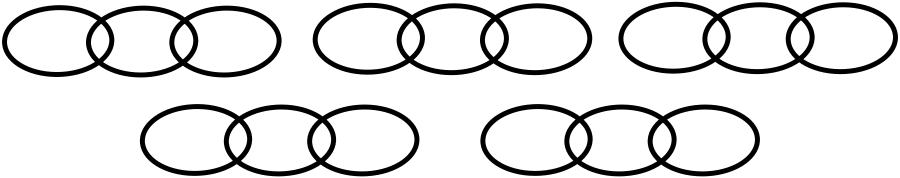
Little Red Riding Hood's grandmother wants to make soup. She needs exactly 4 liters of water, but she only has one jug that holds 5 liters and one that holds 3 liters. Little Red Riding Hood has to run to the well and get the 4 liters of water needed for grandma. How can Little Red Riding Hood do this?

Beard loss, Max von Kleist (FU Berlin) Math calendar 2014 – abbreviated

Seven days until Christmas and Santa Claus has suddenly lost his beard. Of his magnificent beard, consisting of 999 whiskers, only one hair is left. Santa can't perform like this! – What to do? With normal beard growth, 3 beard hairs per day would grow again - that's not enough!
Fortunately, one of the Christmas elves has some dubious contacts and was able to procure a previously untested magical elixir. According to the package insert, the magical elixir suppresses normal beard growth, but at the same time triples the number of existing beard hairs each time it is used. However, it must not be used more than once a day, otherwise you will turn into an Easter bunny!
On which days does Santa Claus have to use the magic elixir to be able to present a magnificent beard with exactly 999 hairs on Christmas (in 7 days)?

Example of backward work with puzzling (∗)

A goldsmith has 5 parts of a chain, each consisting of 3 links. He wants to connect the 5 parts to a (stretched) chain. To open a link costs 30 ct, to close it costs 70 ct.
Can he assemble the chain for 3 euros?



Content specific
Heuristic Strategies

Geometry

- Draw appropriate helping lines,
- Search for equal length lines (isosceles or equilateral triangles, sides of a parallelogram, radii of a circle, . . .),
- Search for equally large or adjacent angles,
- Search for right triangles or partial triangles $\rightsquigarrow$ Pythagoras,
- Search for symmetries or addition to symmetric figures,
- Search for congruent triangles $\rightsquigarrow$ equal distances, equal angles,
- Search for similar triangles $\rightsquigarrow$ distance relations,
- Search for parallel lines $\rightsquigarrow$ angle relations, intersection theorem,
- Search for triangles or quadrilaterals of same area content $\rightsquigarrow$ calculate unknown sizes from area contents,
- Search for surfaces which can be completed to be or split into congruent surfaces.

Teaching of Heuristic Procedures

Teaching heuristic approaches — Development of problem-solving skills

- Teachings through problem solving: Heuristics remain implicit
- Teaching about problem solving: Reflection on heuristics

$\leadsto$ After solving, make the students aware of the procedure

- These were not just "tricks"
- After the solving, recapitulate the procedure on a on a meta level

:-) More and more to be found in schoolbooks

- Use the same heuristics in similar situations on your own.

- Teachers must be able to make difficult tasks accessible to students.
- Teachers must be able to solve difficult tasks.
- Teachers must be able to work through difficult tasks.

Exercise: Making Tasks Accessible

The sides of a rectangle $\square ABCD$ are divided in the ratio 1:2 as shown in the sketch, Let the dividing vertices be (sequentially) E, F, G and H. The intersections of the connecting lines AF, BG, CH and DE form the corners of a quadrilateral $\square PQRS$.

a) What kind of quadrilateral is $\square PQRS$?

b) How does the area of this quadrilateral compare to the area of the rectangle?

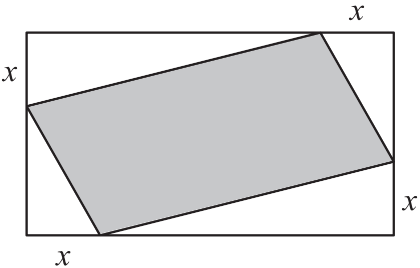
Exercise: Making Tasks Accessible

1. Draw a conjecture for part a) of the task.
2. Find reasons and ideas for proof of your conjecture. Try to find approaches that basically take different paths.
3. What approaches do you think are suitable for teaching?
4. Can your solution be simplified for a special case? Are there any particularities in the transition from the special to the general case?
5. Work out (a) reasonable generalization(s) of this task and try to apply your solutions from part 1 to this generalization(s).
6. What variation(s) of the task and what approaches would you favor for the lesson?

Remark: Dynamic geometry software is useful for the study of special and generalized problems.

Exercise: Making Tasks Accessible ($*$)

- On the sides of a rectangle, a distance of the length x is drawn, each starting from the corner points according to the direction of rotation. The four free end points are connected to form a quadrilateral. For which length x is the area of this quadrilateral minimal?



- Different approaches, special cases, generalizations, ...
- Polya questions about the phases (understanding the task, thinking up a plan, executing the plan, reviewing)

Teaching heuristic approaches — Development of problem-solving skills

- Demonstrate selected procedure (familiarization effect)
- Targeted introduction of the procedure using a striking example (recognition effect).
- Rehearsal using less prominent examples (safety)
- Free application on mixed tasks
- Polya-Questions (content-independent) corresponding to the problem-solving phases (understanding the task, coming up with a plan, executing the plan, reviewing), as a stimulus for teachers to give students impulses
- Also practice the failure.

Promote
creativity

Further approaches to promoting creativity

Teaching conditions for the development of thinking skills

According to E.Ch.Wittmann, *Grundfragen des Mathematikunterrichts*, 1981, S.101f.

1. Acquiring knowledge through discovery learning
2. Encouraging students to think divergently
    - Create a relaxed atmosphere
    - Build self-confidence and courage to take risks
    - Showing interest in non-conforming ideas
    - Constructive handling of mistakes
    - No „You don't have to be afraid!“

3 Disrupt automated thought processes, present apparent paradoxes
Example: A cube has 6 sides, on each side there are 4 edges,
so 6 times 4 edges = 24 edges

Further approaches to promoting creativity

Teaching conditions for the development of thinking skills

4. Pose open and challenging problems
   - Problem is not explicitly formulated
   - Some information has to be obtained first
   - There are several ways to solve the problem
   - The problem can not be solved immediately in one go, but it must be possible to solve it
5. Let the students pose the problems themselves or let them continue
6. Developing a working language
7. Stimulating intuitive reasoning and conjecture
8. Teaching heuristic strategies
9. Establishing a constructive relationship with errors
10. Stimulate discussion, reflection, and argumentation.

Further approaches to promoting creativity

Suggestions for training creativity according to H.Winter, *Entdeckendes Lernen im Mathematikunterricht*, 1991, S.175

1. Do not "give" problems, but develop them out of contexts, that appear challenging, stimulate questioning
2. Opportunities for free experimentation, especially of a sensual nature, which encourage conjecture.
3. Wide learning/discovery aids – fewer result finding aids more aids for finding the result by oneself e.g. Polya questions
4. . . .
5. . . .
6. Talking about thinking, representing, remembering, forgetting, making mistakes, etc.
7. Make the content or "formal" significance of the topic clear