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INNOMATH

Innovative enriching education processes for Mathematically Gifted Students in Europe

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Learning Plan

Topic: Spherical Geometry

Target Group: Students at Grade 7 to 9 (age range: 12-15 years old)/ Form 1 to 3, in a secondary school

Goal/ Content/ Description: In order to help the students comprehend some differences between flat Euclidean geometry and spherical geometry and learn a first way of computing distances and triangular surfaces on a sphere, 3-dimensional models are used and intersections between planes and a sphere are considered.

Then they practice by considering exercises, including ones with real life applications.

Objectives: **What the students will know**:

- To identify shortest connections on a sphere and the plane which will yield this connection by intersecting the sphere.
- To identify the centre angle of a spherical segment.
- To identify the interior angles of a spherical triangle.

What the students will be able to do:

- To calculate arc length at given centre angles.
- To calculate triangular surfaces on a sphere at given interior angles.
- To solve real world problems.
- To state and prove the theorem about how to compute the area of a spherical triangle.
- To develop skills for problem solving.
- To identify/ develop/ create applications of the related concepts and processes in the real world.

What attitudes the approach will foster:

- To develop critical thinking skills.
- To adopt various strategies for problem solving.
- To develop motives and positive affective tendencies for mathematics.
- To exploit experimental approaches to measure large distances on earth.

Materials/ Tools: Traditional board and geometrical equipment. Cardboard, paper, markers, scissors, glue, if possible transparent paper. Small balls and three elastic bands for every student.

Resources used by the teacher:

Spherical Triangles and Girard's Theorem
https://www.math.csi.cuny.edu/abhijit/623/spherical-triangle.pdf
Youtube: Spherical Geometry: Deriving The Formula For The Area Of A Spherical Triangle
https://www.youtube.com/watch?v=Y8VgvoEx7HY

Resources for the student:

Approaches/ Methodology: As a first step the students are asked to explore connecting lines between two given points of a sphere, using a small ball, elastic band and a ruler. They are supposed to look for shortest connections. The students are then asked to specify what these conditions are.

As a second step the students clarify whether sections of latitude or longitude are distance minimising.

Furthermore the students are asked to compute the distance between the north and the south pole and between the north pole and a point on the equator, using the radius of earth and realising that the shortest connections are a semicircle or a quarter circle. This result is then generalised for computing distances between two points on a sphere given the centre angle between them.

Next, students use their newly acquired knowledge to calculate the area between two longitudes, called a spherical digon, by using the angle between them. With the help of a small ball and three elastic bands the students explore how to compute the area of a spherical triangle using digons.

Activities Plan:

Introductory activities (creation of interest, reference to real value issues, relation to background experiences etc)

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
2 weeks earlier than the classroom consideration	STEP 1 **Preparatory work: Provide every student with a small ball and one elastic band.** Explore connecting lines between two given points of a sphere, using a small ball, elastic band and a ruler. Look for shortest connections. Find conditions for a line segment to be the shortest connection between two points. *Solution: Sections of great circles are the shortest connections between two points. Great circles are obtained by intersecting the sphere with a plane containing the centre of the sphere.*	Provide a document with written instructions, a small ball and one elastic band.
1 week earlier than the classroom consideration	Consider the conclusions that the students have reached and proceed to the STEP 2 Take a look at a globe. Which sections are length minimizing, those of longitudes or those of latitudes? Explain your decision.	Provide a document with written instructions.

	Solution: Those of longitudes are length minimizing and those of the equator. Longitudes are created by intersecting the sphere with planes through the north and south poles, which therefore also contain the centre of the sphere. Of the latitudes, only the equator is created by such a cut. Compute the distance between the north and the south pole and between the north pole and a point on the equator (radius of earth approximately 6.38 * 10^6m). *Solution: approximately 20 * 10^6m and 10 * 10^6m (a semicircle and a quarter circle)* Compute the distance between Manaus (Brazil, Amazonas river) and Goose Bay (Canada, Labrador) (radius of earth approximately 6.38 * 10^6m). *Solution: both have longitude 60ºW, Manaus 3,1ºS, Goose Bay 53,3ºN: approximately 56,4/360 * 40 * 10^6m $\approx$ 6,27 * 10^3km (6,2*10^3km is fine)*	

Development activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
In the classroom on the planned day for the lesson	**Preparatory work: Have a globe for demonstration in class. Provide every student with a small ball and an elastic band.** Consider the approaches of the students as they have been developed at home. Discuss the validity and the structure of their presentations Identify difficulties and weaknesses. Generalize by asking to specify : Derive a concrete formula. Specify which sizes are required. Ask them to state it and explain it between themselves. *Solution: $\alpha/2\pi$ = distance/(complete circumference), hence distance = $\alpha/2\pi * 2\pi r = \alpha * r$ (here angles are given in radian measure)*	Discussion How can the distance between two arbitrary points on a sphere be computed?
	Remind the students of the calculation of the circumference of the earth by Erathostenes (or provide a text about it).	Discussion

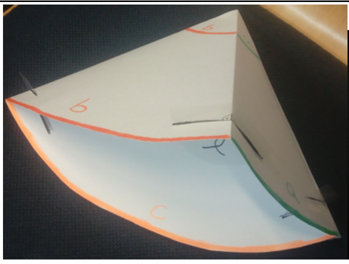
	The students use the small balls and pencils or the like to find solutions. With the help of the globe the solutions can be demonstrated in front of the class. *Solution: The pencils are first placed perpendicularly to the surface of the sphere at the corresponding locations on the globe. Then they are shifted parallel so that the angle between them can be measured.* *Since the rods cannot be shifted parallel during measurements on earth, their angle to the solar radiation is measured at the same time.* In addition, the central angles between different places on earth can be determined using the globe and thus the distances can be calculated. (Examples: Paris – New York: 6283km, Berlin – Dubai: 4620km, Warsaw – Seoul: 7736km)	How can two people at different places on earth determine the distance between them, if they each have only a rod of 1m length, a protractor and a watch that show exactly the same time?

Practicing Activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
At Home	Ask the students to use their newly acquired knowledge to calculate the area between two longitudes, called a spherical digon, by using the angle between them. *Solution: $a/2\pi$ = area/(complete sphere surface), hence area = $a/2\pi * 4\pi r^2 = 2a * r^2$ (here angles are given in radian measure)*	Provide a document with written instructions.
	Computing distances between different places on earth and areas between different longitudes	Provide a document with written instructions.

Development activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
Next day	**Preparatory work: Have a globe for demonstration in class. Provide every student with a small ball and three elastic bands. Make a large model of a spherical triangle including the centre of the sphere and the centre angles from cardboard. Mark the centre angles there. Use transparent paper to mark the inner angles of the spherical triangle.** **An instruction for making an open spherical triangle for students is in the appendix.**	Discussions

	  Consider the approaches of the students as they have been developed at home. Discuss the validity and the structure of their presentations Identify difficulties and weaknesses. Explain, using the model of the spherical triangle, what the central angles and what the inner angles of a spherical triangle are. In particular, discuss with the pupils what possibilities there are for measuring the inner angles of a spherical triangle. • by the angle between the tangents • by the angle between the planes intersecting the sphere along the sides of the triangle Remark: If tangents of circles are not familiar, demonstrate the concept by using a transparent paper as tangent plane to the sphere at the point you are interested in. Then draw the tangents as the lines in the tangent plane which project onto the sides of the triangle, when you look at them “from above”. With the help of a small ball and three elastic bands the students explore how to compute the area of a spherical triangle using digons. *Solution: For explanations see* • *https://www.math.csi.cuny.edu/abhijit/623/spherical-triangle.pdf* • *https://www.youtube.com/watch?v=Y8VgvoEx7HY*	How can the area of a spherical triangle be determined with the help of digons?

Assessment activities

Time When / length	Description of the activity	Instructions/ Hints/ Support/ Comments
	Provide material that will help in realizing the achievement of the objectives: multiple choice test covering both cognitive and affective domain issues Self-assessment	

Appendix

Instructions for making an open spherical triangle for students

Cut out a circle from paper, the center must be marked. Draw four radii that include different center angles (e.g. 50, 70, 100 and 140). Cut out one sector of the circle. Mark the arcs of the remaining circle sectors in different colors. Fold the paper along the radii and glue the two cut radii together. The colored circle arcs now form a spherical triangle. You can mark the inner angles of the spherical triangle by gluing strips of tape over two circle arcs right next to the corners of the triangle so that it does not deform. You can now write the name of the interior angle on the tape.