Derivation of the (natural) exponential function (STEP 1)

- **Problem:** What are the derivative functions of the exponential functions we know?

- **Graphical exploration using Geogebra:** *qualitative*
  Use Geogebra file 1 to describe a graphical qualitative relationship between the exponential functions and their derivative functions.
  a) The derivative function of $g(x) = 2^x$ is ____
  ____
  b) The derivative function of $h(x) = 4^x$ is ____
  ____

  Assumption 1: There must exist an exponential function p for which ______________________________
  ____

- **Graphical exploration using Geogebra:** *quantitative*
  Use Geogebra file 1 and the following tables of values to describe an algebraic quantitative relationship between the exponential functions and their derivative functions.

x	-3	-2	-1	0	1	2	3
$g(x)$							
$g'(x)$							
$\frac{g'(x)}{g(x)}$							
$h(x)$							
$h'(x)$							
$\frac{h'(x)}{h(x)}$							

Assumption 2:
$g(x) = 2^x \quad \Rightarrow \quad g'(x) = \ldots$
$h(x) = 4^x \quad \Rightarrow \quad h'(x) = \ldots$
$f(x) = a^x \quad \Rightarrow \quad f'(x) = \ldots$

- **Confirm the assumption 2:**
  - Differential quotient of $f(x) = a^x$ at the point x_0:

$$f'(x_0) = \lim_{h\to 0}\frac{\qquad - f(x_0)}{\qquad} = \lim_{h\to 0}\frac{\qquad - a^{x_0}}{\qquad} = \lim_{h\to 0}\frac{\qquad - a^{x_0}}{\qquad} = a^{x_0}\cdot\lim_{h\to 0}\frac{\qquad - \qquad}{\qquad}$$

  - Differential quotient of $f(x) = a^x$ at the point 0:

$$f'(0) = \lim_{h\to 0}\frac{\qquad - f(0)}{\qquad} = \lim_{h\to 0}\frac{\qquad - a^{0}}{\qquad} = \lim_{h\to 0}\frac{\qquad - \qquad}{\qquad}$$

  Conclusion: This confirms our assumption and it is generally valid for $f(x) = a^x$: ____________________

- **Confirm the assumption 1:**
  For $f(x) = f'(x)$, it must applied $f'(0) = \ldots$.

$$f'(0) = \lim_{h\to 0}\frac{\qquad - \qquad}{\qquad} = \lim_{n\to\infty}\frac{\qquad - \qquad}{\qquad} = 1 \quad \Leftrightarrow \quad \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} \xrightarrow{n\to\infty} 1 \quad \Leftrightarrow \quad a^{\frac{1}{n}} - 1 \xrightarrow{n\to\infty}$$

$$\Leftrightarrow \qquad\qquad\qquad \Leftrightarrow \quad a \xrightarrow{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

n	1	10	100	1000	10000	100000
a						

Definition: The positive number a for which the exponential function f with $f(x) = a^x$ coincides with its derivative function f' is called **Euler's number** e with $e \approx \ldots$
The corresponding exponential function f with $f(x) = e^x$ is called **natural exponential function**, for its derivative function consequently applies $f'(x) = e^x$.

Derivation rules for exponential functions (STEP 2)

- **Problem:** How to derive e-functions?
- The **elementary derivation rules** (still) apply:
  - **Sum rule:** $f(x) = e^x + 3x^2 \quad \Rightarrow \quad f'(x) =$ ____________
  - **Constant rule:** $f(x) = e^x - 7 \quad \Rightarrow \quad f'(x) =$ ____________
  - **Factor rule:** $f(x) = 4 \cdot e^x \quad \Rightarrow \quad f'(x) =$ ____________
- The **higher derivation rules** apply:
  - **Chain rule** for the concatenation of two functions:

$$f(x) = g(h(x)) \quad \Rightarrow \quad f'(x) = g'(h(x)) \cdot h'(x)$$

(„outer multiplied by inner derivative“)

$g(x)$ – outer function $\qquad g'(x)$ – outer derivative

$h(x)$ – inner function $\qquad h'(x)$ – inner derivative

- **Example 1:** $f(x) = (x^3 + 4)^7$

outer function: $g(x) = x^7$ $\qquad$ outer derivative: $g'(x) =$ ____________

inner funktion: $h(x) =$ ____________ $\qquad$ inner derivative: $h'(x) =$ ____________

The outer derivative with the inner function is: $g'(h(x)) =$ ____________

The inner derivative still follows multiplicatively: $h'(x) =$ ____________

$$\Rightarrow \qquad f'(x) = g'\big(h(x)\big) \cdot h'(x) = \underbrace{\qquad\qquad}_{\text{outer derivative}} \cdot \underbrace{\qquad}_{\text{inner derivative}} = 21x^2 \cdot (x^3 + 4)^6$$

- **Example 2:** $f(x) = e^{-2x^3}$

outer function: $g(x) = e^x$ $\qquad$ outer derivative: $g'(x) =$ ____________

inner funktion: $h(x) =$ ____________ $\qquad$ inner derivative: $h'(x) =$ ____________

The outer derivative with the inner function is: $g'(h(x)) =$ ____________

The inner derivative still follows multiplicatively: $h'(x) =$ ____________

$$\Rightarrow \qquad f'(x) = g'\big(h(x)\big) \cdot h'(x) = \underbrace{\qquad}_{\text{outer derivative}} \cdot \underbrace{\qquad}_{\text{inner derivative}} = -6x^2 \cdot e^{-2x^3}$$

Exercises: Form the first derivative in each case and name the corresponding rules:

a) $f(x) = e^{3x}$

b) $f(x) = 7 \cdot e^{-5x}$

c) $f(x) = 10 + 50 \cdot e^{-0,25x}$

d) $f(x) = S - c \cdot e^{-kx}$

e) $f(x) = (1 + 4 \cdot e^{-2x})^{-1}$

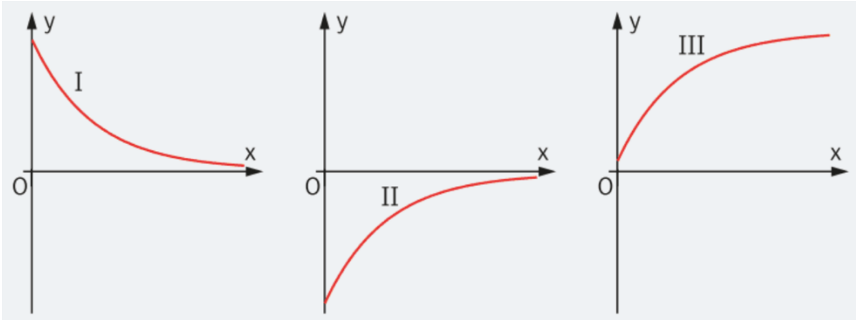
f) $f(x) = -\frac{2}{7+3\cdot e^{-5x}}$

g) $f(x) = \frac{S}{1+a\cdot e^{-kx}}$

Modelling constrained and logistic growth (STEP 3)

- **Problem:** How can processes with constrained growth be modelled?

- Derivation of **constrained growth**:
  - **a)** Describe which geometric operations transform the graph I into graph III.
  - **b)** Graph I describes an exponential function f with the equation $f(x) = c \cdot e^{k \cdot x}$. Characterize the parameters c and k.
    (for help: use Geogebra file 2)
  - **c)** Find an equation for the function with graph III.



Definition: A **constrained growth** with limit S can be described with a continuous function f for $x \in \mathbb{R}$:

$$f(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Describe and explain the change of a population with constrained growth with a function $p(n)$.
  - **a)** $S = 100;\quad p(0) = 10;\quad p(1) = 20$
  - **b)** $S = 100;\quad p(0) = 200;\quad p(10) = 150$
  - **c)** $p(0) = 10;\quad p(1) = 20;\quad p(10) = 70$
    (Here you need a computer algebra system (CAS) for solving)

- **Market development**
  A company launches a new detergent on the market. The marketing department estimates that the company can achieve a market share of 30% with it. After only one month, the market share rises from 0% to 10%.
  - **a)** Analyse whether one can conclude from this, assuming constrained growth, that the market share is already over 25% after half a year.
  - **b)** Determine the rate of increase of the market share after half a year.

- Derivation of **logistic growth**:
  Match each function equation with the correct graph and interpret the influence of the parameters.
  a) $f(x) = \frac{10}{1+8 \cdot e^{-0.8x}}$ **b)** $f(x) = \frac{10}{1+8 \cdot e^{-0.5x}}$ **c)** $f(x) = \frac{10}{1+4 \cdot e^{-0.5x}}$ **d)** $f(x) = \frac{10}{1+4 \cdot e^{-0.4x}}$

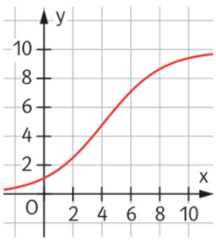


Fig. 1

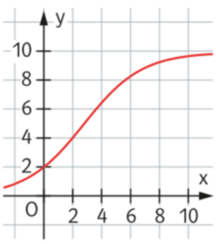


Fig. 2

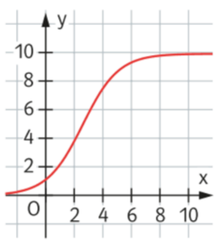


Fig. 3

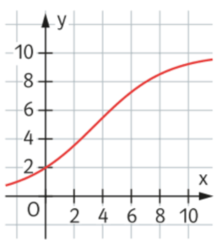
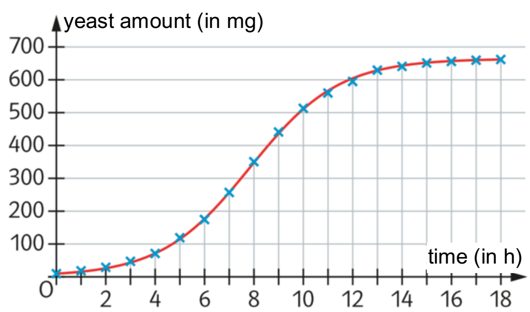


Fig. 4

Definition: A **logistic growth** with limit S can be described with a continuous function f for $x \in \mathbb{R}$:

$$f(x) = \frac{\qquad}{\qquad}$$

- **Yeast growth**
  The researcher Carlson analysed the growth of yeast cultures as early as 1913.
  - **a)** Model the values by a logistic growth function.
  - **b)** Identify justifiably the time at which the yeast culture reaches approximately its maximum growth rate and determine this rate.



Source: Brandt, D. et al. (2016): *Lambacher Schweizer Mathematik. Oberstufe mit CAS-Einsatz*. 1st edition. Ernst Klett Verlag. Stuttgart, p. 224 – 233.