Derivation of the (natural) exponential function (STEP 1)

- **Problem:** What are the derivative functions of the exponential functions we know?

- **Graphical exploration using Geogebra:** *qualitative*
  Use Geogebra file 1 to describe a graphical qualitative relationship between the exponential functions and their derivative functions.

  a) The derivative function of $g(x) = 2^x$ is *proportional to the graph of g, but compressed by a certain factor.*

  b) The derivative function of $h(x) = 4^x$ is *proportional to the graph of h, but stretched by a certain factor.*

  Assumption 1: There must exist an exponential function p for which *the graph of the derivative function p' and the graph of the function p are identical.*

- **Graphical exploration using Geogebra:** *quantitative*
  Use Geogebra file 1 and the following tables of values to describe an algebraic quantitative relationship between the exponential functions and their derivative functions.

x	-3	-2	-1	0	1	2	3
$g(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g'(x)$	0.086	0.173	0.346	0.693	1.383	2.766	5.545
$\frac{g'(x)}{g(x)}$	0.688	0.692	0.692	0.693	0.692	0.692	0.693
$h(x)$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64
$h'(x)$	0.022	0.086	0.346	1.386	5.55	22.185	88.74
$\frac{h'(x)}{h(x)}$	1.408	1.376	1.384	1.386	1.388	1.387	1.387

Assumption 2:

$$g(x) = 2^x \quad \Rightarrow \quad g'(x) = 0.69 \cdot g(x) = g'(0) \cdot g(x)$$

$$h(x) = 4^x \quad \Rightarrow \quad h'(x) = 1.39 \cdot h(x) = h'(0) \cdot h(x)$$

$$f(x) = a^x \quad \Rightarrow \quad f'(x) = k \cdot f(x) = f'(0) \cdot f(x)$$

- **Confirm the assumption 2:**
  - Differential quotient of $f(x) = a^x$ at the point x_0:

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0}\frac{a^{x_0+h} - a^{x_0}}{h} = \lim_{h\to 0}\frac{a^{x_0}\cdot a^h - a^{x_0}}{h} = \underbrace{a^{x_0}}_{f(x_0)} \cdot \lim_{h\to 0}\underbrace{\frac{a^h - 1}{h}}_{f'(0)}$$

  - Differential quotient of $f(x) = a^x$ at the point 0:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{a^h - a^0}{h} = \lim_{h\to 0}\frac{a^h - 1}{h}$$

Conclusion: This confirms our assumption and it is generally valid for $f(x) = a^x$: $\underline{f'(x) = f(x)\cdot f'(0)}$

- **Confirm the assumption 1:**

For $f(x) = f'(x)$, it must applied $f'(0) = 1$.

$$f'(0) = \lim_{h\to 0}\frac{a^h - 1}{h} = \lim_{n\to\infty}\frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1 \quad \Leftrightarrow \quad \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} \xrightarrow{n\to\infty} 1 \quad \Leftrightarrow \quad a^{\frac{1}{n}} - 1 \xrightarrow{n\to\infty} \frac{1}{n}$$

$$\Leftrightarrow \quad a^{\frac{1}{n}} \xrightarrow{n\to\infty} 1 + \frac{1}{n} \quad \Leftrightarrow \quad a \xrightarrow{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

n	1	10	100	1000	10000	100000
a	2	2.59374	2.70481	2.71692	2.71815	2.71827

Definition: The positive number a for which the exponential function f with $f(x) = a^x$ coincides with its derivative function f' is called **Euler's number e** with $e \approx 2.718281828$.
The corresponding exponential function f with $f(x) = e^x$ is called **natural exponential function**, for its derivative function consequently applies $f'(x) = e^x$.

Derivation rules for exponential functions (STEP 2)

- **Problem:** How to derive e-functions?
- The **elementary derivation rules** (still) apply:
  - **Sum rule:** $f(x) = e^x + 3x^2 \quad \Rightarrow \quad f'(x) = e^x + 6x$
  - **Constant rule:** $f(x) = e^x - 7 \quad \Rightarrow \quad f'(x) = e^x$
  - **Factor rule:** $f(x) = 4\cdot e^x \quad \Rightarrow \quad f'(x) = 4\cdot e^x$
- The **higher derivation rules** apply:
  - **Chain rule** for the concatenation of two functions:

$$f(x) = g(h(x)) \quad \Rightarrow \quad f'(x) = g'(h(x))\cdot h'(x)$$

(„outer multiplied by inner derivative“)

$g(x)$ – outer function $\qquad g'(x)$ – outer derivative
$h(x)$ – inner function $\qquad h'(x)$ – inner derivative

- **Example 1:** $f(x) = (x^3+4)^7$

outer function: $g(x) = x^7$ $\qquad$ outer derivative: $g'(x) = 7x^6$

inner funktion: $h(x) = x^3 + 4$ $\qquad$ inner derivative: $h'(x) = 3x^2$

The outer derivative with the inner function is: $g'(h(x)) = 7\cdot(x^3+4)^6$

The inner derivative still follows multiplicatively: $h'(x) = 3x^2$

$$\Rightarrow \qquad f'(x) = g'\big(h(x)\big)\cdot h'(x) = \underbrace{7\cdot(x^3+4)^6}_{\text{outer derivative}} \cdot \underbrace{3x^2}_{\text{inner derivative}} = 21x^2\cdot(x^3+4)^6$$

- **Example 2:** $f(x) = e^{-2x^3}$

outer function: $g(x) = e^x$ $\qquad$ outer derivative: $g'(x) = e^x$

inner funktion: $h(x) = -2x^3$ $\qquad$ inner derivative: $h'(x) = -6x^2$

The outer derivative with the inner function is: $g'(h(x)) = e^{-2x^3}$

The inner derivative still follows multiplicatively: $h'(x) = -6x^2$

$$\Rightarrow \qquad f'(x) = g'\big(h(x)\big)\cdot h'(x) = \underbrace{e^{-2x^3}}_{\text{outer derivative}} \cdot \underbrace{-6x^2}_{\text{inner derivative}} = -6x^2\cdot e^{-2x^3}$$

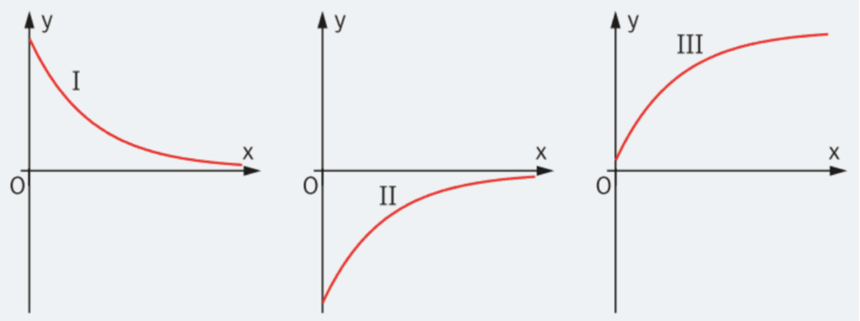
Exercises: Form the first derivative in each case and name the corresponding rules:

a) $f(x) = e^{3x}$	$f'(x) = 3e^{3x}$	**chain rule**
b) $f(x) = 7\cdot e^{-5x}$	$f'(x) = -35\cdot e^{-5x}$	**factor and chain rule**
c) $f(x) = 10 + 50\cdot e^{-0{,}25x}$	$f'(x) = -12{,}5\cdot e^{-0{,}25x}$	**sum, constant, factor and chain rule**
d) $f(x) = S - c\cdot e^{-kx}$	$f'(x) = ck\cdot e^{-kx}$	**sum, constant, factor and chain rule**
e) $f(x) = (1 + 4\cdot e^{-2x})^{-1}$	$f'(x) = \frac{8\cdot e^{-2x}}{(1+4\cdot e^{-2x})^2}$	**chain, sum, constant and factor rule**
f) $f(x) = -\frac{2}{7+3\cdot e^{-5x}} = -2\cdot(7+3\cdot e^{-5x})^{-1}$	$f'(x) = -\frac{30\cdot e^{-5x}}{(7+3\cdot e^{-5x})^2}$	**factor, chain, sum and constant rule**
g) $f(x) = \frac{S}{1+a\cdot e^{-kx}} = S\cdot\left(1 + a\cdot e^{-kx}\right)^{-1}$	$f'(x) = \frac{akS\cdot e^{-kx}}{\left(1+a\cdot e^{-kx}\right)^2}$	**factor, chain, sum and constant rule**

Modelling constrained and logistic growth (STEP 3)

- **Problem:** How can processes with constrained growth be modelled?

- Derivation of **constrained growth**:



a) Describe which geometric operations transform the graph I into graph III.

Graph I is first reflected on the x-axis so that we obtain graph II. Then graph II is shifted along the y-axis so that we get graph III.

b) Graph I describes an exponential function f with the equation $f(x) = c \cdot e^{k \cdot x}$. Characterize the parameters c and k.

The **parameter c** changes the function f along the y-axis: it is stretched if $1 < |c|$ and it is compressed if $0 < |c| < 1$. If the parameter c is negative, the graph of f is reflected on the x-axis. The parameter c also influences the point of intersection with the y-axis: $P_y(0|c)$.

The **parameter k** changes the function f along the x-axis: it approaches the y-axis more and more on one side if $|k|$ is larger and larger. If the parameter k is positive, $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = 0$ applies. If k is negative, the graph of f is reflected on the y-axis and $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = \infty$ applies.

c) Find an equation for the function with graph III.

$$f(x) = -c \cdot e^{-k \cdot x} + S \quad (\text{for } c, k, S > 0)$$

Definition: A **constrained growth** with limit S can be described with a continuous function f for $x \in \mathbb{R}$:

$$\boldsymbol{f(x) = S - c \cdot e^{-k \cdot x}} \qquad (\text{for } c, k, S \in \mathbb{R}^+)$$

- Describe and explain the change of a population with constrained growth with a function $p(n)$.

a) $S = 100; \quad p(0) = 10; \quad p(1) = 20$

$S = 100 \quad \Rightarrow \quad p(n) = 100 - c \cdot e^{-k \cdot n}$

$p(0) = 10 \quad \Rightarrow \quad 10 = 100 - c \cdot e^0 \quad \Rightarrow \quad c = 90 \quad \Rightarrow \quad p(n) = 100 - 90 \cdot e^{-k \cdot n}$

$p(1) = 20 \quad \Rightarrow \quad 20 = 100 - 90 \cdot e^{-k} \quad \Rightarrow \quad e^{-k} = \frac{8}{9} \quad \Rightarrow \quad -k = \ln\left(\frac{8}{9}\right) \quad \Rightarrow \boldsymbol{p(n) = 100 - 90 \cdot e^{\ln\left(\frac{8}{9}\right) \cdot n}}$

The population shows positive growth with a starting value of 10 and an upper limit of 100.

b) $S = 100; \quad p(0) = 200; \quad p(10) = 150$

$S = 100 \quad \Rightarrow \quad p(n) = 100 - c \cdot e^{-k \cdot n}$

$p(0) = 200 \quad \Rightarrow \quad 200 = 100 - c \cdot e^0 \quad \Rightarrow \quad c = -100 \quad \Rightarrow \quad p(n) = 100 + 100 \cdot e^{-k \cdot n}$

$p(10) = 150 \quad \Rightarrow \quad 150 = 100 + 100 \cdot e^{-10k} \quad \Rightarrow \quad e^{-10k} = \frac{1}{2} \quad \Rightarrow \quad -k = \frac{1}{10} \cdot \ln\left(\frac{1}{2}\right)$

$\Rightarrow \boldsymbol{p(n) = 100 + 100 \cdot e^{\frac{1}{10} \cdot \ln\left(\frac{1}{2}\right) \cdot n} = 100 + 100 \cdot e^{-\frac{ln(2)}{10} \cdot n}}$

The population shows negative growth with a starting value of 200 and a lower limit of 100.

c) $p(0) = 10; \quad p(1) = 20; \quad p(10) = 70$

$p(0) = 10 \quad \Rightarrow \quad 10 = S - c \cdot e^0 \quad \Rightarrow \quad S = c + 10 \quad \Rightarrow \quad p(n) = c + 10 - c \cdot e^{-k \cdot n}$

$p(1) = 20 \quad \Rightarrow \quad 20 = c + 10 - c \cdot e^{-k} \quad \Rightarrow \quad 10 = c\left(1 - e^{-k}\right) \quad \Rightarrow \quad c = \frac{10}{1 - e^{-k}}$

$p(10) = 70 \quad \Rightarrow \quad 70 = c + 10 - c \cdot e^{-10k} \quad \Rightarrow \quad 60 = c\left(1 - e^{-10k}\right) \quad \Rightarrow \quad c = \frac{60}{1 - e^{-10k}}$

equating the two equations: $\frac{10}{1-e^{-k}} = \frac{60}{1-e^{-10k}} \quad \Leftrightarrow \quad 1 - e^{-10k} = 6\left(1 - e^{-k}\right) \quad \Leftrightarrow \quad 6e^{-k} - e^{-10k} = 5$

The CAS provides $k \approx 0.128$ as the solution to the equation.

$\Rightarrow \quad c \approx 83.17 \quad \Rightarrow \quad S \approx 93.17 \quad \Rightarrow \quad \boldsymbol{p(n) = 93.17 - 83.17 \cdot e^{-0.128 \cdot n}}$

The population shows positive growth with a starting value of 10 and an upper limit of 93.17.

- **Market development**

  A company launches a new detergent on the market. The marketing department estimates that the company can achieve a market share of 30% with it. After only one month, the market share rises from 0% to 10%.

  a) Analyse whether one can conclude from this, assuming constrained growth, that the market share is already over 25% after half a year.

  At first we model the equation of the function:

  $\rightarrow f(x) = S - c \cdot e^{-k \cdot x}$ (x time in months, $f(x)$ market share in percent)

  From the text we can gather the following information: $S = 30$; $f(0) = 0$; $f(1) = 10$

  $\rightarrow S = 30 \quad \Rightarrow f(x) = 30 - c \cdot e^{-k \cdot x}$

  $\rightarrow f(0) = 0 \quad \Rightarrow 0 = 30 - c \cdot e^{0} \quad \Rightarrow c = 30 \quad \Rightarrow f(x) = 30 - 30 \cdot e^{-k \cdot x}$

  $\rightarrow f(1) = 10 \quad \Rightarrow 10 = 30 - 30 \cdot e^{-k} \quad \Rightarrow e^{-k} = \frac{2}{3} \quad \Rightarrow -k = \ln\left(\frac{2}{3}\right)$

  $\Rightarrow \boldsymbol{f(x) = 30 - 30 \cdot e^{\ln\left(\frac{2}{3}\right) \cdot x}}$

  Now we determine when the market share is 25%:

  $\rightarrow f(x) = 25 \quad \Rightarrow 30 - 30 \cdot e^{\ln\left(\frac{2}{3}\right) \cdot x} = 25 \quad \Rightarrow e^{\ln\left(\frac{2}{3}\right) \cdot x} = \frac{1}{6} \quad \Rightarrow x = \frac{\ln\left(\frac{1}{6}\right)}{\ln\left(\frac{2}{3}\right)} \approx 4.419$

  This means that by the middle of the 5th month, the market share will already be 25%.

  b) Determine the rate of increase of the market share after half a year.

  For the rate of increase, we need the first derivative: $f'(x) = -30 \cdot \ln\left(\frac{2}{3}\right) \cdot e^{\ln\left(\frac{2}{3}\right) \cdot x}$

  $\Rightarrow f'(6) \approx 1.068$

  After half a year, the rate of increase is about 1.07% per month.

- Derivation of **logistic growth**:

  Match each function equation with the correct graph and interpret the influence of the parameters.

  a) $f(x) = \frac{10}{1+8 \cdot e^{-0.8x}}$ **b)** $f(x) = \frac{10}{1+8 \cdot e^{-0.5x}}$ **c)** $f(x) = \frac{10}{1+4 \cdot e^{-0.5x}}$ **d)** $f(x) = \frac{10}{1+4 \cdot e^{-0.4x}}$

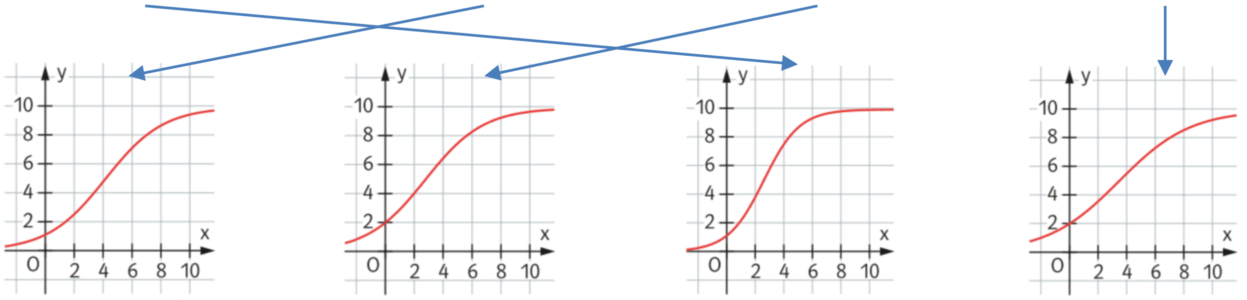


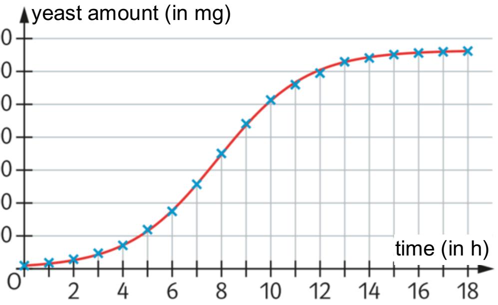
Fig. 1 Fig. 2 Fig. 3 Fig. 4

The numerator describes the upper limit S of growth.
The factor of the exponential function in the denominator influences the point of intersection with the y-axis.
The factor in the exponent of the exponential function influences the speed of growth; it regulates how quickly the limit is reached.

Definition: A **logistic growth** with limit S can be described with a continuous function f for $x \in \mathbb{R}$:

$$\boldsymbol{f(x) = \frac{S}{1 + a \cdot e^{-k \cdot x}}} \qquad (\text{for } a, k, S \in \mathbb{R}^{+})$$

- **Yeast growth**



The researcher Carlson analysed the growth of yeast cultures as early as 1913.

a) Model the values by a logistic growth function.

We use as equation of the function:

$f(x)=\frac{S}{1+a\cdot e^{-k\cdot x}}$ (x time in h, $f(x)$ yeast amount in mg)

and read three suitable values from the figure:

$S=670;\ f(0)=10;\ f(12)=600$

$\rightarrow S=670 \qquad \Rightarrow\ f(x)=\frac{670}{1+a\cdot e^{-k\cdot x}}$

$\rightarrow f(0)=10 \qquad \Rightarrow\ 10=\frac{670}{1+a\cdot e^{0}} \qquad \Rightarrow\ a=66 \qquad \Rightarrow\ f(x)=\frac{670}{1+66\cdot e^{-k\cdot x}}$

$\rightarrow f(12)=600\ \Rightarrow\ 600=\frac{670}{1+66\cdot e^{-12k}} \qquad \Rightarrow\ e^{-12k}=\frac{7}{3960} \qquad \Rightarrow\ -k=\frac{1}{12}\ln\left(\frac{7}{3960}\right)$

$$\Rightarrow\ \boldsymbol{f(x)=\frac{670}{1+66\cdot e^{\frac{\ln\left(\frac{7}{3960}\right)}{12}\cdot x}}\approx f(x)=\frac{670}{1+66\cdot e^{-0.5282\cdot x}}}$$

b) Identify justifiably the time at which the yeast culture reaches approximately its maximum growth rate and determine this rate.

The graph of a function describing logistic growth is approximately point-symmetric. The mean value between the minimum 10 and the maximum 670 is 330 and is reached after about **8 hours**.

We form the derivative function of f with $\boldsymbol{f(x)=670\cdot(1+66\cdot e^{-0.5282\cdot x})^{-1}}$ to determine the growth rate:

$$\boldsymbol{f'(x)=-670\cdot(1+66\cdot e^{-0.5282\cdot x})^{-2}\cdot 66\cdot e^{-0.5282\cdot x}\cdot(-0.5282)}$$

$$\boldsymbol{f'(x)\approx\frac{23357\cdot e^{-0.5282\cdot x}}{(1+66\cdot e^{-0.5282\cdot x})^{2}}}$$

$$\boldsymbol{\rightarrow\ f'(8)\approx 88.44\frac{mg}{h}}$$

The maximum growth rate is about $88.44\frac{mg}{h}$.

Source: Brandt, D. et al. (2016): *Lambacher Schweizer Mathematik. Oberstufe mit CAS-Einsatz*. 1st edition. Ernst Klett Verlag. Stuttgart, p. 224 – 233.